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CUBIC DIFFERENTIAL SYSTEMS WITH INVARIANT LINES OF TOTAL MULTIPLICITY EIGHT

111.02 - Differential Equations

Synopsis

of Doctor Thesis in Mathematics

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The Doctoral Thesis and the Synopsis can be consulted at the Central Library "Andrei Lupan" of the Academy of Sciences of Moldova and on the webpage of the National Council for Accreditation and Attestation (www.cnaa.md).

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THESIS CONCEPTUAL REFERENCE POINTS

Current state of the problem and identification of research problems.
We consider real planar polynomial differential system, i.e. systems of the form

$$\dot{x} = P(x,y), \quad \dot{y} = Q(x,y), \tag{1.1}$$

where P and Q are polynomials in x and y with real coefficients and $\max\big(\deg(P), \deg(Q)\big) = n$. We call such systems *polynomial systems of degree n.*

The existence of invariant algebraic curves in planar polynomial systems has been a relevant element in the study of integrability of these systems. In 1878 Darboux gave the theory of invariant algebraic curves of polynomial differential equations and the integrability of these equations in terms of the invariant curves. A Modern presentation of the the theory of Darboux can be found in [24, 27].

The existence of sufficiently many invariant straight lines of a polynomial system (1.1) could be used for integrability of such systems. During the past 15 years several articles were published on this theme. Investigations concerning polynomial differential systems possessing invariant straight lines were done by Popa, Sibirski, Kooij, Sokulski, Zhang Xi Kang, Schlomiuk, Vulpe, Dai Guo Ren, Artes, Llibre as well as Dolov and Kruglov.

Several authors have dealt with the simplest class of polynomial systems (1.1) (i.e. $n \leq 3$), and namely Popa, Sibirski, Schlomiuk, Vulpe, etc.

In this Thesis are approached the planar differential cubic systems, i.e. $n = 3$. The set $\mathbb{CS}$ of *cubic differential systems* depends on 20 parameters and for this reason people began by studying particular subclasses of $\mathbb{CS}$. We are interested in the investigation of the family of cubic systems with *invariant straight lines* ($\mathbb{ISL}s$). Artes and Llibre [1] showed that the maximum number of invariant straight lines taking into account their multiplicities for a polynomial differential system of degree m is $3m$ when we also consider the infinite straight line. So the maximum number of the invariant straight lines (including the line at infinity $Z = 0$) for cubic systems with non-degenerate infinity is at most 9. A classification of all cubic systems possessing the maximum number of invariant straight lines taking into account their multiplicities has been made in [25]. The authors used the notion of configuration of invariant lines, as introduced in [28], for cubic systems and detected 23 such configurations. Moreover using invariant polynomials with respect to the action of *the group* $Aff(2,\mathbb{R})$ *of affine transformations and time rescaling* in this paper, the necessary and sufficient conditions for the realization of each one of 23 configurations were detected. A new class of cubic systems omitted in [25] was constructed by the author of the thesis in [8].

In the Thesis we continue the investigations started in [25]. More exactly, here we shall consider the family of cubic systems with invariant lines of total multiplicity eight, including the line at infinity and considering their multiplicities (we denote this family by $\mathbb{CSL}_8$). The results concerning these systems are exhibit in [4–22].

Some cubic systems with invariant lines have been also investigated by Lyubimova [26], Şubă, Puţuntică, Repeşco [29, 30] and by other mathematicians (Llibre, Mahdi, Vulpe, Cozma, Chan Guo Wei). Lyubimova considered cubic systems in $\mathbb{CSL}_8$ which possess 7 $\mathbb{IL}s$, all real and distinct, and constructed in this case 4 configurations. Şubă and his students, using the notion of *parallel multiplicity*, arrived at 17 configurations of $\mathbb{IL}s$ which coincide with those obtained in our classification in the case of cubic systems with four *infinite singular points* ($\mathbb{ISP}s$). But in contrast with their work, for each configuration we give the necessary and sufficient conditions for its realization in terms of invariant polynomials with respect to the group of affine transformations and time rescaling. We note that the invariant polynomials was constructed applying the Invariant Theory of Differential Equations, founded by C. Sibirschi and developed by his disciples (Lunchevici, Marinciuc, Gasinschi–Chirniţchi, Dang Dini Bic, Tacu, Vulpe, Popa, Boularas Driss, Baltag, Calin, Daniliuc, etc.).

The purpose and objectives of the thesis. The main goal of the Thesis is to give a full classification for cubic systems with invariant straight lines of total multiplicity eight. This classification involves the realization of the following objectives:

1. to detect all possible configurations of invariant straight lines for this family of systems;

2. to construct necessary and sufficient affine invariant conditions for the realization of each one of the detected configurations.

Methodology of scientific study. The research carried out in is Thesis are based on methods of Qualitative Theory of Dynamical Systems, Invariant Theory of Differential Equation, methods of Bifurcation Theory of Dynamic Systems on a plane, methods of Algebraic Computations.

Novelty and scientific originality. In our Thesis for the first time there are constructed all possible configurations of invariant lines of total multiplicity eight for cubic systems. Our set of configurations contains as particular cases all the configurations detected by other authors for special cases of systems in $\mathbb{CSL}_8$ (see [26], [29,30]). But in contrast with these papers in the Thesis we have constructed necessary and sufficient conditions for the realization of each one of the corresponding configurations. Moreover we detect a new class of cubic systems with invariant lines of total multiplicity nine.

The main scientific problem which is solved in this Thesis consists in classifying the whole family of cubic differential systems possessing invariant lines of total multiplicity eight according to configurations of these lines; this classification is very helpful for obtaining the complete topological classification of this family and is useful for the study of integrability of these systems.

The significance of theoretical and practical values of the work. The results obtained in this thesis for cubic systems with invariant lines of total multiplicity 8 represent a significant step in the algebraic and geometric part of theory of cubic systems.

Principal scientific results to be defended:

(a) all possible 51 configurations of invariant straight lines for cubic systems possessing invariant lines of total multiplicity eight;

(b) the necessary and sufficient affine invariant conditions for the realization of each one of 51 configurations;

(c) the representatives of the family of systems with invariant lines of total multiplicity eight modulo the action of the affine group and time rescaling;

(d) the perturbed canonical systems which characterize the vicinities of cubic systems in $\mathbb{CSL}_8$;

(e) a new class of cubic systems possessing invariant lines of total multiplicity nine which completes the classification given by Llibre and Vulpe in [25].

Implementation of the scientific results. The scientific results obtained could be used for a deeper investigation of cubic systems possessing invariant straight lines of total multiplicity eight (including the line at infinity), and namely:

- the configurations of invariant lines detected, and canonical forms could be used for a complete topological classification of cubic systems in this class;

- the canonical forms constructed for cubic cubic systems in $\mathbb{CSL}_8$ can serve as a basis for determining of the first integrals of such systems;

- the necessary and sufficient affine invariant conditions can be applied for any cubic system in order to detect if it belongs to $\mathbb{CSL}_8$ and if so, then to specify its configuration of invariant lines;

- this classification could be helpful for further investigations of cubic systems with invariant lines of total multiplicity less than 8;

- the scientific results obtained can be applied in the study of some mathematical models which are described by polynomial differential systems and which are related with some problems in physics, chemistry, medicine and so on.

- these investigations could serve as a support for teaching courses in higher education.

Approval of obtained scientific results. The scientific results obtained and to be defined were examined and approved by various research seminars, which are as follows: Qualitative Theory of Differential Equations of Moldova State University, 2015; Differential Equations and Algebras of Tiraspol State University, 2013, 2014; seminar of the Department of Differential Equations and Systems Analysis of Belorussian State University, Minsk, 2013; seminar of the Department of Mathematics of the Shanghai Normal University, Shanghai (China), 2015.

Main scientific results included in the Thesis were presented at several scientific conferences: International Conference of Young Researchers, X-th Edition, Chişinău, 2012; Conference on Applied and Industrial Mathematics (CAIM), Chişinău: U.S.T., 2012, 2014, 2015; International Conference "Mathematics and Information Technologies: Research and Education" (MITRE), Chişinău: U.S.M., 2013-2015; Conferinţa Ştiinţifică Internaţională a doctoranzilor "Tendinţe Contemporane ale Dezvoltării Ştiinţei: Viziuni ale Tinerilor Cercetători", Chişinău: AŞM, 2014, 2015; The Third Conference of Mathematical Society of Moldova (IMCS-50), Chişinău: AŞM, 2014; Conferinţa Ştiinţifică Internaţională cu participare internaţională "Probleme actuale ale ştiinţelor exacte şi ale naturii", Chişinău: U.S.T., 2015.

Research papers. Research outcomes are reflected in 19 publications: 3 preprints, 6 scientific peer-reviewed articles (including 4 in ISI journals), 10 proceedings and abstracts of international conferences; 2 articles and 4 abstracts are published as single-author papers.

Keywords: cubic differential system, group of affine transformations, invariant polynomial, invariant straight line, multiplicity of a line, configuration of invariant straight lines, type of configuration, canonical form, perturbed system..

The language of the Thesis is English. The base text comprises 154 pages and has the following structure: Introduction, 4 Chapters, General Conclusions and Recommendations, Bibliography with 140 References and 28 figures.

CONTENTS OF THE THESIS

The **Introduction** reveals the actual status of the conducted research, main reasons for carrying on the proposed research, the purpose and objectives of the thesis, the importance and advantages of the conducted scientific investigations, novelty and scientific originality, scientific and research problems solved, the scientific results to be defended, as well as the approval of obtained scientific results.

Chapter 1 contains a survey of the most important results related to the purpose and objectives of the Thesis. In the fist section we give a brief survey on cubic differential systems with invariant straight lines. More exactly, we discuss about the qualitative theory of differential systems and the importance to study the configurations of invariant lines for cubic systems which serve as a a basis for completing the phase portraits of the corresponding systems. So, our Thesis was partly motivated by the problem of topologically classifying the cubic differential systems. In the second section we describe the problem of integrability concerning differential systems (1.1). Having obtained all canonical forms for cubic systems possessing invariant lines of total multiplicity eight, the problem of integrability of such systems could be resolved and this also motivated our work. The last section is devoted to the concept of invariant polynomial and its use in classification problems. We briefly review the classical theory of invariants and its analog for the theory of polynomial vector fields developed by Sibirskii school and its new developments by the joint work of the Chişinău school, the Barcelona school and by Schlomiuk.

In **Chapter 2** we firstly give the preliminary definitions and results needed in the work. In this chapter we give the preliminary definitions and results needed in the work. This section is devoted to some aspects concerning the Invariant Theory and besides some invariant polynomials earlier constructed we exhibit 52 new invariant polynomials, which are in fact $CT-$comitants.

Consider real differential cubic systems, i.e. systems of the form:

$$(S) \quad \dot{x} = p_0 + p_1(x,y) + p_2(x,y) + p_3(x,y), \quad \dot{y} = q_0 + q_1(x,y) + q_2(x,y) + q_3(x,y) \qquad (1)$$

Let $f,\, g \in \mathbb{R}[a,x,y]$ and $(f,g)^{(k)} = \sum_{h=0}^{k}(-1)^h\binom{k}{h}\dfrac{\partial^k f}{\partial x^{k-h}\partial y^h}\;\dfrac{\partial^k g}{\partial x^h \partial y^{k-h}}$. The polynomial $(f,g)^{(k)} \in \mathbb{R}[a,x,y]$ is called the *transvectant of index k of the polynomials f and g.*

In order to define the needed invariant polynomials we first construct the following comitants of second degree with respect to the coefficients of the initial system:

$$
\begin{array}{llll}
S_1 = (C_0, C_1)^{(1)}, & S_8 = (C_1, C_2)^{(2)}, & S_{15} = (C_2, D_2)^{(1)}, & S_{22} = (D_2, D_3)^{(1)}, \\
S_2 = (C_0, C_2)^{(1)}, & S_9 = (C_1, D_2)^{(1)}, & S_{16} = (C_2, C_3)^{(1)}, & S_{23} = (C_3, C_3)^{(2)}, \\
S_3 = (C_0, D_2)^{(1)}, & S_{10} = (C_1, C_3)^{(1)}, & S_{17} = (C_2, C_3)^{(2)}, & S_{24} = (C_3, C_3)^{(4)}, \\
S_4 = (C_0, C_3)^{(1)}, & S_{11} = (C_1, C_3)^{(2)}, & S_{18} = (C_2, C_3)^{(3)}, & S_{25} = (C_3, D_3)^{(1)}, \\
S_5 = (C_0, D_3)^{(1)}, & S_{12} = (C_1, D_3)^{(1)}, & S_{19} = (C_2, D_3)^{(1)}, & S_{26} = (C_3, D_3)^{(2)}, \\
S_6 = (C_1, C_1)^{(2)}, & S_{13} = (C_1, D_3)^{(2)}, & S_{20} = (C_2, D_3)^{(2)}, & S_{27} = (D_3, D_3)^{(2)}. \\
S_7 = (C_1, C_2)^{(1)}, & S_{14} = (C_2, C_2)^{(2)}, & S_{21} = (D_2, C_3)^{(1)}, &
\end{array}
$$

We shall use here the following invariant polynomials constructed in [25] to characterize the family of cubic systems possessing the maximal number (i.e. nine) of invariant straight lines: $\mathcal{D}_1(a) = 6S_{24}^3 - \big[(C_3, S_{23})^{(4)}\big]^2$, $\mathcal{D}_2(a,x,y) = -S_{23}$, $\mathcal{D}_3(a,x,y) = (S_{23},\, S_{23})^{(2)} - 6C_3(C_3,\, S_{23})^{(4)}$, $\mathcal{D}_4(a) = (C_3,\, \mathcal{D}_2)^{(4)}$, $\mathcal{V}_1(a,x,y) = S_{23} + 2D_3^2$, $\mathcal{V}_2(a,x,y) = S_{26}$, $\mathcal{V}_3(a,x,y) = 6S_{25} - 3S_{23} - 2D_3^2$, $\mathcal{V}_4(a,x,y) = C_3\left[(C_3, S_{23})^{(4)} + 36\,(D_3, S_{26})^{(2)}\right]$, $\mathcal{L}_1(a,x,y) = 9C_2\,(S_{24} + 24S_{27}) - 12D_3\,(S_{20} + 8S_{22}) - 12\,(S_{16}, D_3)^{(2)} - 3\,(S_{23}, C_2)^{(2)} - 16\,(S_{19}, C_3)^{(2)} + 12\,(5S_{20} + 24S_{22}, C_3)^{(1)}$, $\mathcal{L}_2(a,x,y) = 32\,(13\,S_{19} + 33\,S_{21}, D_2)^{(1)} + 84\,(9\,S_{11} - 2\,S_{14}, D_3)^{(1)} - 448\,(S_{18}, C_2)^{(1)} + 8D_2\,(12S_{22} + 35S_{18} - 73S_{20}) - 56\,(S_{17}, C_2)^{(2)} - 63\,(S_{23}, C_1)^{(2)} + 756D_3S_{13} - 1944D_1S_{26} + 112\,(S_{17}, D_2)^{(1)} - 378\,(S_{26}, C_1)^{(1)} + 9C_1\,(48S_{27} - 35S_{24})$, $\mathcal{U}_1(a) = S_{24} - 4S_{27}$, $\mathcal{U}_2(a,x,y) = 6(S_{23} - 3S_{25}, S_{26})^{(1)} - 3S_{23}(S_{24} - 8S_{27}) - 24S_{26}^2 + 2C_3(C_3, S_{23})^{(4)} + 24D_3(D_3, S_{26})^{(1)} + 24D_3^2S_{27}$.

However these invariant polynomials are not sufficient to characterize the cubic systems with invariant lines of total multiplicity 8. So we constructed here the following new invariant polynomials:

$$
\begin{aligned}
\mathcal{V}_5(a,x,y) &= 6T_1(9A_5 - 7A_6) + 2T_2(4T_{16} - T_{17}) - 3T_3(3A_1 + 5A_2) + 3A_2T_4 + 36T_5^2 - 3T_{44}, \\
\mathcal{V}_6(a,x,y) &= 6D_3^2 + S_{23} + 6S_{25}, \quad \mathcal{L}_6(a) = 2A_3 - 19A_4, \quad \mathcal{L}_7(a,x,y) = (T_{10}, T_{10})^{(2)}, \\
\mathcal{K}_1(a,x,y) &= \big(3223T_2^2T_{140} + 2718T_4T_{140} - 829T_2^2T_{141}, T_{133}\big)^{(10)}/2, \quad \mathcal{K}_2(a,x,y) = T_{74}, \\
\mathcal{K}_3(a,x,y) &= Z_1Z_2Z_3, \quad \mathcal{K}_4(a,x,y) = T_{13} - 2T_{11}, \\
\mathcal{K}_5(a,x,y) &= 45T_{42} - T_2T_{14} + 2T_2T_{15} + 12T_{36} + 45T_{37} - 45T_{38} + 30T_{39}, \\
\mathcal{K}_6(a,x,y) &= 4T_1T_8(2663T_{14} - 8161T_{15}) + 6T_8(178T_{23} + 70T_{24} + 555T_{26}) + \\
&\quad 18T_9(30T_2T_8 - 488T_1T_{11} - 119T_{21}) + 5T_2(25T_{136} + 16T_{137}) - \\
&\quad 15T_1(25T_{140} - 11T_{141}) - 165T_{142}, \quad \mathcal{K}_7(a) = A_1 + 3A_2, \\
\mathcal{K}_8(a,x,y) &= 10A_4T_1 - 3T_2T_{15} + 4T_{36} - 8T_{37}, \; \mathcal{K}_9(a,x,y) = 3T_1(11T_{15} - 8T_{14}) - T_{23} + 5T_{24},
\end{aligned}
$$

$$
\begin{aligned}
N_1(a,x,y) &= S_{13}, \quad N_2(a,x,y) = T_9, \quad N_3(a,x,y) = C_2D_3 + 3S_{16},\\
N_4(a,x,y) &= -S_{14}^2 - 2D_2^2(3S_{14} - 8S_{15}) - 12D_3(S_{14}, C_1)^{(1)} +\\
&\quad + D_2(-48D_3S_9 + 16(S_{17}, C_1)^{(1)}),\\
N_5(a,x,y) &= 36D_2D_3(S_8 - S_9) + D_1(108D_2^2D_3 - 54D_3(S_{14} - 8S_{15})) +\\
&\quad + 2S_{14}(S_{14} - 22S_{15}) - 8D_2^2(3S_{14} + S_{15}) - 9D_3(S_{14}, C_1)^{(1)} - 16D_2^4,\\
N_6(a,x,y) &= 40D_3^2(15S_6 - 4S_3) - 480D_2D_3S_9 - 20D_1D_3(S_{14} - 4S_{15}) +\\
&\quad + 160D_2^2S_{15} - 35D_3(S_{14}, C_1)^{(1)} + 8\big((S_{23}, C_2)^{(1)}, C_0\big)^{(1)},\\
N_7(a,x,y) &= 18C_2D_2(9D_1D_3 - S_{14}) - 2C_1D_3(8D_2^2 - 3S_{14} - 74S_{15}) -\\
&\quad - 432C_0D_3S_{21}48S_7(8D_2D_3 + S_{17}) + 6S_{10}(12D_2^2 + 151S_{15}) -\\
&\quad - 51S_{10}S_{14} - 162D_1D_2S_{16} + 864D_3(S_{16}, C_0)^{(1)},\\
N_8(a,x,y) &= -32D_3^2S_2 - 108D_1D_3S_{10} + 108C_3D_1S_{11} - 18C_1D_3S_{11} - 27S_{10}S_{11} +\\
&\quad + 4C_0D_3(9D_2D_3 + 4S_{17}) + 108S_4S_{21},\\
N_9(a,x,y) &= 11S_{14}^2 - 16D_1D_3(16D_2^2 + 19S_{14} - 152S_{15}) - 8D_2^2(7S_{14} + 32S_{15}) -\\
&\quad - 2592D_1^2S_{25} + 88D_2(S_{14}, C_2)^{(1)},\\
N_{10}(a,x,y) &= -24D_1D_3 + 4D_2^2 + S_{14} - 8S_{15},\\
N_{11}(a,x,y) &= S_{14}^2 + D_1[16D_2^2D_3 - 8D_3(S_{14} - 8S_{15})] - 2D_2^2(5S_{14} - 8S_{15}) +\\
&\quad + 8D_2(S_{14}, C_2)^{(1)},\\
N_{12}(a,x,y) &= -160D_2^4 - 1620D_3^2S_3 + D_1(1080D_2^2D_3 - 135D_3(S_{14} - 20S_{15})) -\\
&\quad - 5D_2^2(39S_{14} - 32S_{15}) + 85D_2(S_{14}, C_2)^{(1)} + 81\big((S_{23}, C_2)^{(1)}, C_0\big)^{(1)} + 5S_{14}^2,\\
N_{13}(a,x,y) &= 2(136D_3^2S_2 - 126D_2D_3S_4 + 60D_2D_3S_7 + 63S_{10}S_{11}) -\\
&\quad - 18C_3D_1(S_{14} - 28S_{15}) - 12C_1D_3(7S_{11} - 20S_{15}) - 192C_2D_2S_{15} +\\
&\quad + 4C_0D_3(21D_2D_3 + 17S_{17}) + 3C_2(S_{14}, C_2)^{(1)},\\
N_{14}(a,x,y) &= -6D_1D_3 - 15S_{12} + 2S_{14} + 4S_{15},\\
N_{15}(a,x,y) &= 216D_1D_3(63S_{11} - 104D_2^2 - 136S_{15}) + 4536D_3^2S_6 + 4096D_2^4 +\\
&\quad + 120S_{14}^2 + 992D_2(S_{14}, C_2)^{(1)} - 135D_3\big[28(S_{17}, C_0)^{(1)} + 5(S_{14}, C_1)^{(1)}\big],\\
N_{16}(a,x,y) &= 2C_1D_3 + 3S_{10}, \;\; N_{17}(a,x,y) = 6D_1D_3 - 2D_2^2 - (C_3, C_1)^{(2)},\\
N_{18}(a,x,y) &= 2D_2^3 - 6D_1D_2D_3 - 12D_3S_5 + 3D_3S_8,
\end{aligned}
$$

$$N_{19}(a,x,y) = C_1D_3(18D_1^2-S_6)+C_0(4D_2^3-12D_1D_2D_3-18D_3S_5+9D_3S_8)+6C_2D_1S_8+$$
$$+2\big(9D_2D_3S_1-4D_2^2S_2+12D_1D_3S_2-9C_3D_1S_6-9D_3(S_4,C_0)^{(1)}\big),$$
$$N_{20}(a,x,y) = 3D_2^4-8D_1D_2^2D_3-8D_3^2S_6-16D_1D_3S_{11}+16D_2D_3S_9,$$
$$N_{21}(a,x,y) = 2D_1D_2^2D_3-4D_3^2S_6+D_2D_3S_8+D_1(S_{23},C_1)^{(1)},$$
$$N_{22}(a,x,y) = T_8, \quad N_{23}(a,x,y)=T_6, \quad N_{24}(a,x,y)=2T_3T_{74}-T_1T_{136},$$
$$N_{25}(a,x,y) = 5T_3T_6-T_1T_{23},\ N_{26}=9T_{135}-480T_6T_8-40T_2T_{74}-15T_2T_{75},$$
$$N_{27}(a,x,y) = 9T_2T_9(2T_{23}-5T_{24}-80T_{25})+144T_{25}(T_{23}+5T_{24}+15T_{26})-$$
$$-9(T_{23}^2-5T_{24}^2-33T_9T_{76}), \quad N_{28}(a,x,y)=T_3+T_4,$$
$$W_1(a,x,y) = 2C_2D_3-3C_3D_2,$$
$$W_2(a,x,y) = 6C_3(S_{12}+6S_{11})-9C_1(S_{23}+S_{25})-8(S_{16},C_2)^{(1)}-C_3D_2^2,$$
$$W_3(a,x,y) = 12D_1C_3-S_{10},\ W_4(a,x,y)=-27S_4+4S_7,$$
$$W_5(a,x,y) = 3D_1^2C_1+4D_1S_2-3(S_4,C_0)^{(1)},$$
$$W_6(a,x,y) = 2C_2D_1+3S_4,\ W_7(a,x,y)=(S_{10},D_2)^{(1)},$$
$$W_8(a,x,y) = 4C_2(27D_1D_3-8D_2^2)+2C_2(20S_{15}-4S_{14}+39S_{12})+18C_1(3S_{21}-D_2D_3)+$$
$$+54D_3(3S_4-S_7)-288C_3S_9+54(S_7,C_3)^{(1)}-567(S_4,C_3)^{(1)}+135C_0D_3^2,$$
$$W_9(a,x,y) = 3S_6D_2^2+4S_3D_2^2-6D_1D_2S_9,$$
$$W_{10}(a,x,y) = 18D_1^2C_2+15S_6C_2-6D_1C_1D_2+4C_0D_2^2+27D_1S_4-6C_1S_9,$$
$$W_{11}(a,x,y) = 9C_0D_3^5-6D_3^4(C_1D_2-S_7)+4C_2D_3^3(D_2^2+S_{14}-2S_{15})-$$
$$-12C_3D_3^2[5D_2S_{14}-4D_2S_{15}-7(S_{14},C_2)^{(1)}],$$
$$W_{12}(a,x,y) = -480T_6T_8+9T_{135}-40T_2T_{74}-15T_2T_{75},$$

where

$$Z_1 = 2C_1D_2D_3-9C_0(S_{25}+2D_3^2)+4C_2(9D_1D_3+S_{14})-3C_3(6D_1D_2+5S_8)+36D_3S_4,$$
$$Z_2 = 12D_1S_{17}+2D_2(3S_{11}-2S_{14})+6D_3(S_8-6S_5)-9(S_{25},C_0)^{(1)},$$
$$Z_3 = 48D_1^3C_3+12D_1^2(C_1D_3-C_2D_2)+36D_1(C_0S_{17}-C_3S_6)-16D_2^2S_2-16S_2S_{14}+$$
$$+2C_0D_2(3S_{11}+2S_{14})+3D_3(8D_2S_1+3C_0S_8-2C_1S_6)-9S_4S_8$$
$$-216C_3(S_5,C_0)^{(1)})+6C_2(D_2S_6-4(S_{14},C_0)^{(1)})+54D_1D_2(S_4+D_3C_0).$$

Here the polynomials

$$\begin{aligned}
A_1 &= S_{24}/288, \quad A_2 = S_{27}/72, \; A_3 = (72D_1A_2 + \big(S_{22}, D_2\big)^{(1)})/24,\\
A_4 &= \big[9D_1(S_{24}-288A_2)+4\big(9S_{11}-2S_{14}, D_3\big)^{(2)}+8\big(3S_{18}-S_{20}-4S_{22}, D_2\big)^{(1)}\big]/2^7/3^3,\\
A_5 &= \big(S_{23}, C_3\big)^{(4)}/2^7/3^5, \quad A_6 = \big(S_{26}, D_3\big)^{(2)}/2^5/3^3
\end{aligned}$$

are affine invariants, whereas the polynomials

$$\begin{aligned}
T_1 &= C_3, \quad T_2 = D_3, \quad T_3 = S_{23}/18, \quad T_4 = S_{25}/6, \quad T_5 = S_{26}/72,\\
T_6 &= \big[3C_1(D_3^2 - 9T_3 + 18T_4) - 2C_2(2D_2D_3 - S_{17} + 2S_{19} - 6S_{21}) +\\
&\quad +2C_3(2D_2^2 - S_{14} + 8S_{15})\big]/2^4/3^2,\\
T_8 &= \big[5D_2(D_3^2 + 27T_3 - 18T_4) + 20D_3S_{19} + 12\big(S_{16}, D_3\big)^{(1)} - 8D_3S_{17}\big]/5/2^5/3^3,\\
T_9 &= \big[9D_1(9T_3-18T_4-D_3^2)+2D_2(D_2D_3-3S_{17}-S_{19}-9S_{21})+18\big(S_{15}, C_3\big)^{(1)}-\\
&\quad -6C_2(2S_{20} - 3S_{22}) + 18C_1S_{26} + 2D_3S_{14}\big]/2^4/3^3,\\
T_{11} &= \big[\big(D_3^2 - 9T_3 + 18T_4, C_2\big)^{(2)} - 6\big(D_3^2 - 9T_3 + 18T_4, D_2\big)^{(1)} - 12\big(S_{26}, C_2\big)^{(1)} +\\
&\quad +12D_2S_{26} + 432(A_1 - 5A_2)C_2\big]/2^7/3^4,\\
T_{13} &= \big[27(T_3, C_2)^{(2)} - 18(T_4, C_2)^{(2)} + 48D_3S_{22} - 216(T_4, D_2)^{(1)} + 36D_2S_{26} -\\
&\quad -1296C_2A_1 - 7344C_2A_2 + (D_3^2, C_2)^{(2)}\big]/2^7/3^4,\\
T_{14} &= \big[\big(8S_{19} + 9S_{21}, D_2\big)^{(1)} - D_2(8S_{20} + 3S_{22}) + 18D_1S_{26} + 1296C_1A_2\big]/2^4/3^3,\\
T_{15} &= 8\big(9S_{19} + 2S_{21}, D_2\big)^{(1)} + 3\big(9T_3 - 18T_4 - D_3^2, C_1\big)^{(2)} - 4\big(S_{17}, C_2\big)^{(2)} +\\
&\quad +4\big(S_{14} - 17S_{15}, D_3\big)^{(1)} - 8\big(S_{14} + S_{15}, C_3\big)^{(2)} + 432C_1(5A_1 + 11A_2) +\\
&\quad +36D_1S_{26} - 4D_2(S_{18} + 4S_{22})\big]/2^6/3^3,\\
T_{21} &= \big(T_8, C_3\big)^{(1)}, \quad T_{23} = \big(T_6, C_3\big)^{(2)}/6, \quad T_{24} = \big(T_6, D_3\big)^{(1)}/6,\\
T_{26} &= \big(T_9, C_3\big)^{(1)}/4, \quad T_{30} = \big(T_{11}, C_3\big)^{(1)}, \quad T_{31} = \big(T_8, C_3\big)^{(2)}/24,\\
T_{32} &= \big(T_8, D_3\big)^{(1)}/6, \quad T_{36} = \big(T_6, D_3\big)^{(2)}/12, \quad T_{37} = \big(T_9, C_3\big)^{(2)}/12,\\
T_{38} &= \big(T_9, D_3\big)^{(1)}/12, \quad T_{39} = \big(T_6, C_3\big)^{(3)}/2^4/3^2, \quad T_{42} = \big(T_{14}, C_3\big)^{(1)}/2,\\
T_{44} &= \big((S_{23}, C_3)^{(1)}, D_3\big)^{(2)}/5/2^6/3^3,\\
T_{74} &= \big[27C_0(9T_3 - 18T_4 - D_3^2)^2 + C_1\big(- 62208T_{11}C_3 - 3(9T_3 - 18T_4 - D_3^2) \times\\
&\quad \times(2D_2D_3 - S_{17} + 2S_{19} - 6S_{21})\big) + 20736T_{11}C_2^2 + C_2(9T_3 - 18T_4 - D_3^2) \times\\
&\quad \times(8D_2^2+54D_1D_3-27S_{11}+27S_{12}-4S_{14}+32S_{15})-54C_3(9T_3-18T_4-D_3^2) \times\\
&\quad \times(2D_1D_2 - S_8 + 2S_9) - 54D_1(9T_3 - 18T_4 - D_3^2)S_{16} -\\
&\quad -576T_6(2D_2D_3 - S_{17} + 2S_{19} - 6S_{21})\big]/2^8/3^4, \quad T_{133} = (T_{74}, C_3)^{(1)},
\end{aligned}$$

$$T_{136} = \big(T_{74}, C_3\big)^{(2)}/24, \quad T_{137} = \big(T_{74}, D_3\big)^{(1)}/6, \quad T_{140} = \big(T_{74}, D_3\big)^{(2)}/12,$$
$$T_{141} = \big(T_{74}, C_3\big)^{(3)}/36, \quad T_{142} = \big((T_{74}, C_3)^{(2)}, C_3\big)^{(1)}/72$$

are the elements of the polynomial basis of T-comitants up to degree six for systems (1) constructed by Iu. Calin [23].

Consider the differential operator $\mathcal{L} = x \cdot \mathbf{L}_2 - y \cdot \mathbf{L}_1$ [3] acting on $\mathbb{R}[a, x, y]$, where $\mathbf{L}_1 = 3a_{00}\frac{\partial}{\partial a_{10}} + 2a_{10}\frac{\partial}{\partial a_{20}} + a_{01}\frac{\partial}{\partial a_{11}} + \frac{1}{3}a_{02}\frac{\partial}{\partial a_{12}} + \frac{2}{3}a_{11}\frac{\partial}{\partial a_{21}} + a_{20}\frac{\partial}{\partial a_{30}} + 3b_{00}\frac{\partial}{\partial b_{10}} + 2b_{10}\frac{\partial}{\partial b_{20}} + b_{01}\frac{\partial}{\partial b_{11}} + \frac{1}{3}b_{02}\frac{\partial}{\partial b_{12}} + \frac{2}{3}b_{11}\frac{\partial}{\partial b_{21}} + b_{20}\frac{\partial}{\partial b_{30}}$, $\mathbf{L}_2 = 3a_{00}\frac{\partial}{\partial a_{01}} + 2a_{01}\frac{\partial}{\partial a_{02}} + a_{10}\frac{\partial}{\partial a_{11}} + \frac{1}{3}a_{20}\frac{\partial}{\partial a_{21}} + \frac{2}{3}a_{11}\frac{\partial}{\partial a_{12}} + a_{02}\frac{\partial}{\partial a_{03}} + 3b_{00}\frac{\partial}{\partial b_{01}} + 2b_{01}\frac{\partial}{\partial b_{02}} + b_{10}\frac{\partial}{\partial b_{11}} + \frac{1}{3}b_{20}\frac{\partial}{\partial b_{21}} + \frac{2}{3}b_{11}\frac{\partial}{\partial b_{12}} + b_{02}\frac{\partial}{\partial b_{03}}$.

Using this operator and the affine invariant $\mu_0 = \text{Resultant}_x\big(p_3(a, x, y), q_3(a, x, y)\big)/y^9$ we construct the following polynomials: $\mu_i(a, x, y) = \frac{1}{i!}\mathcal{L}^{(i)}(\mu_0)$, $i = 1, .., 9$, where $\mathcal{L}^{(i)}(\mu_0) = \mathcal{L}(\mathcal{L}^{(i-1)}(\mu_0))$ and $\mathcal{L}^{(0)}(\mu_0) = \mu_0$.

These invariant polynomials are responsible for the total multiplicity of all finite singularities of a cubic system [2, 3]. Moreover all these polynomials vanish for a cubic system if and only if this system is degenerate.

In this chapter we also describe the scheme of the proofs of the main theorems.
Let $L(x, y) = Ux + Vy + W = 0$ be an invariant straight line of cubic systems (S). By its definition we have $UP(x, y) + VQ(x, y) \equiv (Ux + Vy + W)(Ax^2 + 2Bxy + Cy^2 + Dx + Ey + F)$, and this identity provides the following 10 relations: $Eq_1 = (a_{30} - A)U + b_{30}V = 0$, $Eq_2 = (3a_{21} - 2B)U + (3b_{21} - A)V = 0$, $Eq_3 = (3a_{12} - C)U + (3b_{12} - 2B)V = 0$, $Eq_4 = a_{03}U + (b_{03} - C)V = 0$, $Eq_5 = (a_{20} - D)U + b_{20}V - AW = 0$, $Eq_6 = (2a_{11} - E)U + (2b_{11} - D)V - 2BW = 0$, $Eq_7 = a_{02}U + (b_{02} - E)V - CW = 0$, $Eq_8 = (a_{10} - F)U + b_{10}V - DW = 0$, $Eq_9 = a_{01}U + (b_{01} - F)V - EW = 0$, $Eq_{10} = a_{00}U + b_{00}V - FW = 0$.

According to [28] we call **configuration of invariant straight lines** of a system (1.1), the set of (complex) invariant straight lines (which may have real coefficients) of the system, each endowed with its own multiplicity and together with all the real singular points of this system located on these invariant straight lines, each one endowed with its own multiplicity.

If a cubic system is in $\mathbb{CSL}_8$ we say that its lines form a *configuration of type* $(3, 2, 1, 1)$ if there exist one triplet and one couple of parallel lines and two additional lines every set with different slopes. In a similar way are defined *configurations of types* $(3, 3, 1)$, $(3, 2, 2)$ and $(2, 2, 2, 1)$. These four types of the configurations exhaust all possible configurations formed by 8 invariant lines for a cubic system. Note that in all configurations the invariant straight line which is omitted is the infinite one.

Following [25, Lemma 5] we split the family of cubic systems $\mathbb{CSL}_8$ in 9 subfamilies according to the number of infinite singularities (real or complex) of systems (S) which are determined by the linear factors of the polynomial $C_3(x,y) = yp_3(x,y) - xq_3(x,y)$. For each one of these subfamilies the proof of the corresponding Main Theorem proceeds in the following steps:

(i) First we construct the cubic homogeneous parts $(\tilde{P}_3, \tilde{Q}_3)$ of systems for which the corresponding necessary conditions, provided by Theorem 1.3 of the Thesis are satisfied in order to have a given number of triplets or/and of couples of invariant parallel lines in the respective directions;

(ii) Secondly, taking cubic systems $\dot{x} = \tilde{P}_3,\ \dot{y} = \tilde{Q}_3$ we add all quadratic, linear and constant terms and using the equations $Eq_1 - Eq_{10}$ we determine these terms in order to get the necessary number of invariant lines in the respective configuration. Thus the second step ends with the construction of the canonical systems possessing the needed configuration;

(iii) The third step consists in the determination of the affine invariant conditions necessary and sufficient for a cubic system to belong to the family of systems (constructed at the second step) which possess the corresponding configuration of invariant lines;

(iv) And finally, in the case of the existence of multiple invariant lines in a potential configuration we construct the corresponding perturbed systems possessing 8 distinct invariant lines (including the line at infinity).

In section 2.2 of Chapter 2 we state and prove the classification theorem (Main Theorem A) of cubic systems in $\mathbb{CSL}_8$ having four $\mathbb{ISP}s$ according to their configurations of invariant lines and for each configuration we give the corresponding necessary and sufficient conditions in terms of algebraic invariants and comitants with respect to the group of affine transformations and time rescaling. For the family of such cubic systems we also construct its representatives modulo the action of the group under consideration.

Main Theorem A. *Assume that a non-degenerate cubic system (i.e. $\sum_{i=0}^{9} \mu_i^2 \neq 0$) possesses invariant lines of total multiplicity 8, including the line at infinity with its own multiplicity. In addition we assume that this system has four distinct infinite singularities. Then:*

I. *The system possesses exactly one of the 17 possible configurations Config. 8.1 – Config. 8.17 of invariant lines given in Figure 2.1;*

II. *This system possesses the specific configuration Config. 8.j ($j \in \{1, 2, \ldots, 17\}$ if and only if the corresponding conditions included below are fulfilled. Moreover the system can be brought via an affine transformation and time rescaling to the canonical forms, written below* *next to the configuration:*

1) *Four real distinct infinite singularities* $\Leftrightarrow \mathcal{D}_1 > 0,\ \mathcal{D}_2 > 0,\ \mathcal{D}_3 > 0:$

A_1) *Configuration of type* $(3,3,1)$ $\Leftrightarrow \mathcal{V}_1 = \mathcal{V}_2 = \mathcal{L}_1 = \mathcal{L}_2 = \mathcal{K}_1 = 0,\ \mathcal{K}_2 \neq 0;$

- *Config. 8.1* $\Leftrightarrow$ $\mathcal{K}_3 > 0$: $\dot{x} = x(x+1)(x-a),\ \dot{y} = y(y+1)(y-a),\ 0<a\neq 1;$
- *Config. 8.2* $\Leftrightarrow$ $\mathcal{K}_3 < 0$: $\dot{x} = x\big[(x+a)^2+1\big],\ \dot{y} = y\big[(y+a)^2+1\big],\ a\neq 0;$
- *Config. 8.3* $\Leftrightarrow$ $\mathcal{K}_3 = 0$: $\dot{x} = x^2(1+x),\quad \dot{y} = y^2(1+y);$

A_2) *Configuration of type* $(3,2,1,1)$ $\Leftrightarrow \mathcal{V}_5 = \mathcal{U}_2 = \mathcal{K}_4 = \mathcal{K}_5 = \mathcal{K}_6 = 0,\quad \mathcal{D}_4 \neq 0:$

- *Config. 8.4* $\Leftrightarrow$ $\mathcal{L}_1 \neq 0$ *and* $\mathcal{K}_7 > 0$: $\begin{cases} \dot{x} = x(x-1)(x+r), & r>0, \\ \dot{y} = y(y-1)\big[(1-r)x + ry + r\big]; \end{cases}$
- *Config. 8.5* $\Leftrightarrow$ $\mathcal{L}_1 \neq 0$ *and* $\mathcal{K}_7 < 0$: $\begin{cases} \dot{x} = x(x-1)(x+r), & r<0, \\ \dot{y} = y(y-1)\big[(1-r)x + ry + r\big]; \end{cases}$
- *Config. 8.6* $\Leftrightarrow$ $\mathcal{L}_1 = 0$: $\dot{x} = rx^3,\quad \dot{y} = (r-1)xy^2 + y^3,\ r\neq 0;$

A_3) *Configuration of type* $(2,2,2,1)$ $\Leftrightarrow \mathcal{V}_3 = \mathcal{K}_2 = \mathcal{K}_4 = \mathcal{K}_8 = 0,\quad \mathcal{D}_4 \neq 0:$

- *Config. 8.7* $\Leftrightarrow$ $\mathcal{K}_9 > 0$: $\begin{cases} \dot{x} = (x^2-1)(rx+2y+ry),\ r(r^2-1)\neq 0, \\ \dot{y} = (y^2-1)(x+2rx+y), (r+2)(2r+1)\neq 0; \end{cases}$
- *Config. 8.8* $\Leftrightarrow$ $\mathcal{K}_9 < 0$: $\begin{cases} \dot{x} = (x^2+1)(rx+2y+ry),\ r(r^2-1)\neq 0, \\ \dot{y} = (y^2+1)(x+2rx+y), (r+2)(2r+1)\neq 0; \end{cases}$
- *Config. 8.9* $\Leftrightarrow$ $\mathcal{K}_9 = 0$: $\begin{cases} \dot{x} = x^2(rx+2y+ry),\ r(r^2-1)\neq 0, \\ \dot{y} = y^2(x+2rx+y),\ (r+2)(2r+1)\neq 0; \end{cases}$

2) *Two real and two complex distinct infinite singularities* $\Leftrightarrow \mathcal{D}_1 < 0$:

A_4) *Configuration of type* $(3,3,1)$ $\Leftrightarrow \mathcal{V}_1 = \mathcal{V}_2 = \mathcal{L}_1 = \mathcal{L}_2 = \mathcal{K}_1 = 0,\ \mathcal{K}_2 \neq 0;$

- *Config. 8.10* $\Leftrightarrow$ $\mathcal{K}_3 > 0$: $\begin{cases} \dot{x} = (1-r^2)x/4 + x^2 - y^2 + x^3 - 3xy^2, \\ \dot{y} = (1-r^2)y/4 + 2xy + 3x^2y - y^3, r^2\neq 0, 1/9,\ 1; \end{cases}$
- *Config. 8.11* $\Leftrightarrow$ $\mathcal{K}_3 < 0$: $\begin{cases} \dot{x} = (1+r^2)x/4 + x^2 - y^2 + x^3 - 3xy^2, \\ \dot{y} = (1+r^2)y/4 + 2xy + 3x^2y - y^3, \quad r\neq 0; \end{cases}$
- *Config. 8.12* $\Leftrightarrow$ $\mathcal{K}_3 = 0$: $\begin{cases} \dot{x} = x/4 + x^2 - y^2 + x^3 - 3xy^2, \\ \dot{y} = y/4 + 2xy + 3x^2y - y^3; \end{cases}$

A_5) *Configuration of type* $(3,2,1,1)$ $\Leftrightarrow \mathcal{V}_5 = \mathcal{U}_2 = \mathcal{K}_4 = \mathcal{K}_5 = \mathcal{K}_6 = 0,\quad \mathcal{D}_4 \neq 0:$

- *Config. 8.13* $\Leftrightarrow$ $\mathcal{L}_1 \neq 0$: $\begin{cases} \dot{x} = (1+r^2)x\big[(x+r)^2+1\big], \quad r\neq 0, \\ \dot{y} = (1+r^2)^2y + 2r(1+r^2)xy - rx^3 \\ \qquad + r^2x^2y - rxy^2 - y^3; \end{cases}$

- *Config. 8.14* $\Leftrightarrow$ $\mathcal{L}_1 = 0$: $\begin{cases} x = (1+r^2)x^3, \quad r \neq 0, \\ \dot{y} = -rx^3 + r^2x^2y - rxy^2 - y^3; \end{cases}$

A_6) *Configuration of type* $(2,2,2,1)$ $\Leftrightarrow \mathcal{V}_3 = \mathcal{K}_2 = \mathcal{K}_4 = \mathcal{K}_8 = 0, \quad \mathcal{D}_4 \neq 0:$

- *Config. 8.15* $\Leftrightarrow$ $\mathcal{K}_9 > 0$: $\begin{cases} \dot{x} = x(x-1)(1+r^2-2x+2ry), \quad r \neq 0, \\ \dot{y} = -(1+r^2)y + (3+r^2)xy - rx^3 \\ \qquad -3x^2y - 2ry^2 + rxy^2 - y^3; \end{cases}$

- *Config. 8.16* $\Leftrightarrow$ $\mathcal{K}_9 < 0$: $\begin{cases} \dot{x} = 2(1+x^2)(ry - x - r), \ r \neq 0, \\ \dot{y} = r(r^2+3)x + (1-r^2)y - rx^3 \\ \qquad -3x^2y + rxy^2 - y^3; \end{cases}$

- *Config. 8.17* $\Leftrightarrow$ $\mathcal{K}_9 = 0$: $\begin{cases} \dot{x} = -2x^2(x - ry), \ r \neq 0, \\ \dot{y} = -2ry^2 - rx^3 - 3x^2y + rxy^2 - y^3; \end{cases}$

III. *This system could not have a configuration of invariant lines of the type* $(3,3,2)$ *and neither could it have 4 complex* $\mathbb{ISP}s$.

Remark 2.3. If in a configuration an invariant straight line has multiplicity $k > 1$, then the number k appears near the corresponding straight line and this line is in bold face. Real invariant straight lines are represented by continuous lines, whereas complex invariant straight lines are represented by dashed lines. We indicate next to the real singular points of the system, located on the invariant lines, their corresponding multiplicities. In order to describe the various kinds of multiplicity for $\mathbb{ISP}s$ we use the notation (a,b). By this notation we point out the maximum number a (respectively b) of infinite (respectively finite) singularities which can be obtained by perturbation of the multiple point.

Chapter 3 is devoted to the proofs of two classification theorems: Main Theorem B which deals with cubic systems in $\mathbb{CSL}_8$ possessing three distinct $\mathbb{ISP}s$ and Main Theorem C which is related to cubic systems in $\mathbb{CSL}_8$ possessing exactly one $\mathbb{ISP}$. So we detect all possible configurations of $\mathbb{ISL}s$ for these subfamilies of systems and for each configuration we give the necessary and sufficient conditions for its realization. In order to prove the multiplicity of multiple lines we construct in this case the perturbed systems corresponding to the given canonical forms associated to the configurations of invariant lines.

Main Theorem B. *Assume that a non-degenerate cubic system (i.e.* $\sum_{i=0}^{9} \mu_i^2 \neq 0$*) possesses invariant straight lines of total multiplicity 8, including the line at infinity with its own multiplicity. In addition we assume that this system has three distinct infinite singularities, i.e. the conditions* $\mathcal{D}_1 = 0$ *and* $\mathcal{D}_3 \neq 0$ *hold. Then:*

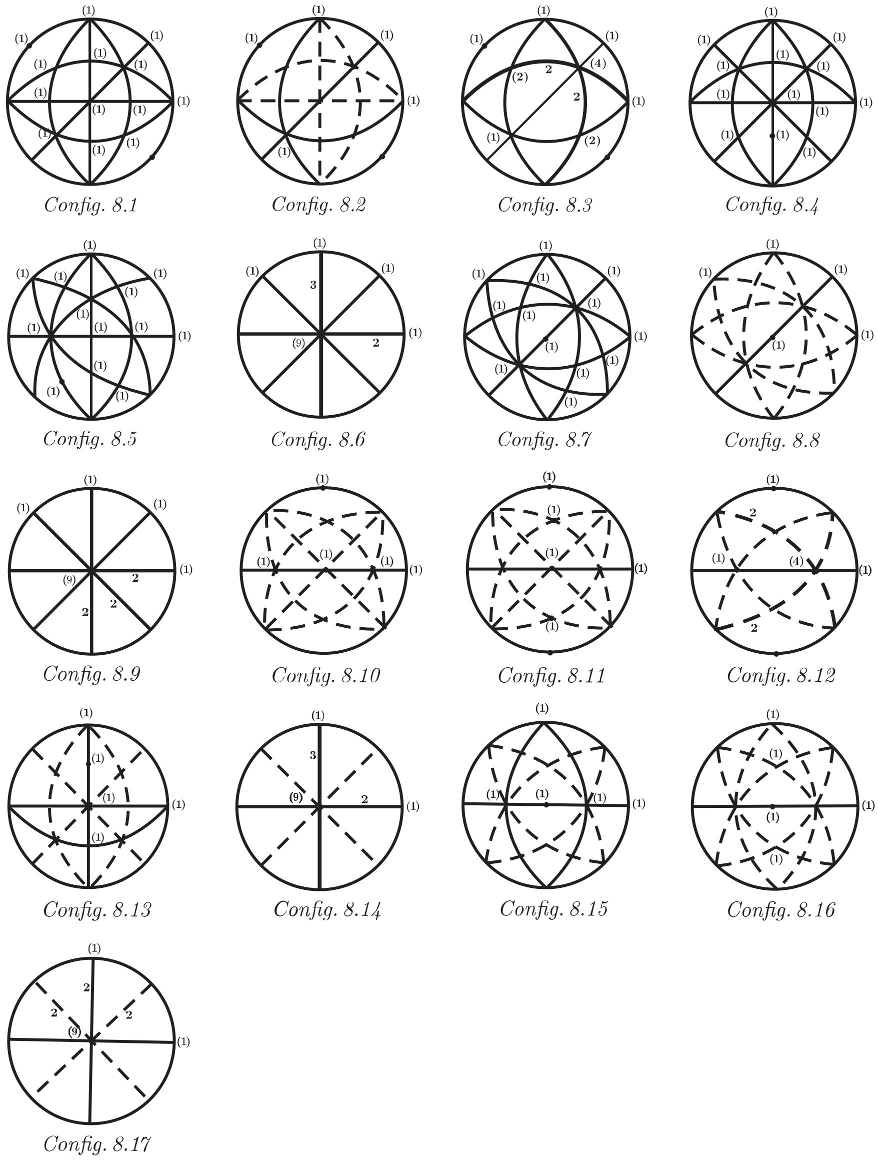


Fig. 2.1. Configurations of invariant lines for systems in $\mathbb{CSL}_8$ with 4 $\mathbb{ISP}s$

I. *This system has only real infinite singularities and it possesses one of the five possible configurations Figure 3.1, endowed with the corresponding conditions included below. Moreover the system could be brought via an affine transformation and time rescaling to the canonical forms, written below next to the configurations;*

II. *This system could not have a configuration of invariant lines of the types* $(3,3,1)$ *or* $(3,2,2)$. *And this system has:*

B_1) *Configuration of type* $(3,2,1,1)$ $\Leftrightarrow \mathcal{V}_4 = \mathcal{V}_5 = \mathcal{K}_4 = \mathcal{K}_5 = \mathcal{K}_6 = 0$;

- *Config. 8.18* $\Leftrightarrow$ $\mathcal{K}_7 \neq 0$, $\mathcal{L}_1 \neq 0$: $\begin{cases} \dot{x} = x(x^2 - 9x - xy - y^2), \\ \dot{y} = -y^2(9 + y); \end{cases}$

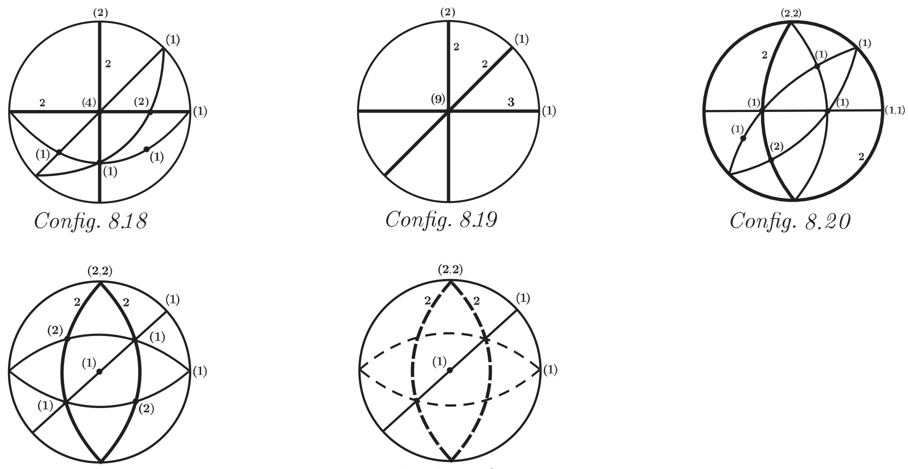
- *Config. 8.19* $\Leftrightarrow$ $\mathcal{K}_7 \neq 0$, $\mathcal{L}_1 = 0$: $\begin{cases} \dot{x} = x(x^2 - xy - y^2), \\ \dot{y} = -y^3; \end{cases}$

- *Config. 8.20* $\Leftrightarrow$ $\begin{matrix} \mathcal{K}_7 = 0,\ \mathcal{L}_1 \neq 0, \\ \mathcal{V}_5 = \mathcal{L}_6 = 0 \end{matrix}$: $\begin{cases} \dot{x} = (1-x)x(1+y), \\ \dot{y} = y(1 - x + y - x^2); \end{cases}$

B_2) Configuration of type $(2,2,2,1)$ $\Leftrightarrow \mathcal{V}_3 = \mathcal{K}_4 = \mathcal{K}_2 = \mathcal{K}_8 = 0, \quad \mathcal{L}_7 \neq 0;$

- *Config. 8.21* $\Leftrightarrow$ $\mathcal{K}_9 > 0$: $\begin{cases} \dot{x} = (x^2 - 1)(x + y), \\ \dot{y} = 2x(y^2 - 1); \end{cases}$

- *Config. 8.22* $\Leftrightarrow$ $\mathcal{K}_9 < 0$: $\begin{cases} \dot{x} = (1 + x^2)(x + y), \\ \dot{y} = 2x(1 + y^2). \end{cases}$



Config. 8.18 *Config. 8.19* *Config. 8.20*

Config. 8.21 *Config. 8.22*

Fig. 3.1. Configurations of invariant lines for systems in $\mathbb{CSL}_8$ with 3 $\mathbb{ISP}s$

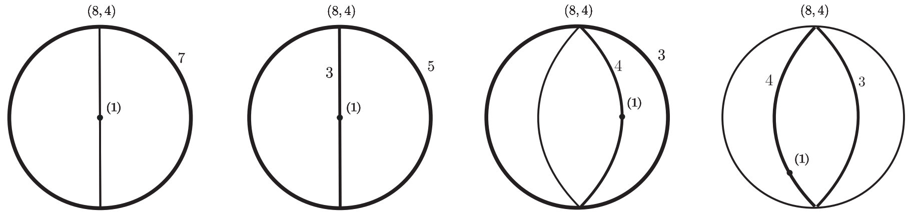
Main Theorem C. *Assume that a non-degenerate cubic system (i.e. $\sum_{i=0}^{9} \mu_i^2 \neq 0$) possesses invariant straight lines of total multiplicity 8, including the line at infinity with its own multiplicity. In addition we assume that this system has exactly one infinite singularity defined by a unique real factor of degree four of $C_3(x,y)$, i.e. the conditions $\mathcal{D}_1 = \mathcal{D}_2 = \mathcal{D}_3 = 0$ hold. Then this system possesses the specific configuration Config. 8.j ($j \in \{48, \ldots, 51\}$) (see Figure 3.2) if and only if the corresponding conditions included below are fulfilled. Moreover it can be brought via an affine transformation and time rescaling to the canonical form, written below next to the conditions:*

Config. 8.48 $\Leftrightarrow$ $\begin{bmatrix} \mathcal{V}_1 = \mathcal{L}_2 = N_{23} = W_1 = W_2 = \\ = W_3 = W_4 = 0 \end{bmatrix}$ $\Leftrightarrow$ $\begin{cases} \dot{x} = x, \\ \dot{y} = -2y - x^3; \end{cases}$

$$\text{Config. } 8.49 \Leftrightarrow \left[\begin{array}{l} \mathcal{V}_1 = \mathcal{L}_2 = N_{23} = W_1 = W_2 = \\ = N_{16} = W_5 = 0, \; W_6 \neq 0 \end{array}\right] \Leftrightarrow \begin{cases} \dot{x} = x, \\ \dot{y} = y - x^2 - x^3; \end{cases}$$

$$\text{Config. } 8.50 \Leftrightarrow \left[\begin{array}{l} \mathcal{V}_1 = \mathcal{L}_2 = N_3 = W_7 = W_8 = \\ = W_9 = W_{10} = 0, \; N_{23} \neq 0 \end{array}\right] \Leftrightarrow \begin{cases} \dot{x} = x(1+x), \\ \dot{y} = y + xy - x^3; \end{cases}$$

$$\text{Config. } 8.51 \Leftrightarrow \left[\begin{array}{l} \mathcal{V}_5 = \mathcal{K}_4 = \mathcal{K}_5 = \mathcal{K}_8 = \mathcal{K}_9 = \\ = N_2 = \mathcal{K}_6 = W_{11} = W_{12} = 0, \\ \mathcal{V}_1 \neq 0 \end{array}\right] \Leftrightarrow \begin{cases} \dot{x} = x^2(1+x), \\ \dot{y} = -1 - 3x + x^2 y - x^3. \end{cases}$$



Config. 8.48 *Config. 8.49* *Config. 8.50* *Config. 8.51*

Fig. 3.2. Configurations of invariant lines for systems in $\mathbb{CSL}_8$ with one $\mathbb{ISP}$

In **Chapter 4** we state and prove the classification theorem (Main Theorem D) of the cubic systems in $\mathbb{CSL}_8$ with two distinct $\mathbb{ISP}s$. For this family of systems we construct all possible configurations of invariant lines and invariant criteria for the realization of each one of the detected configuration. In order to prove the multiplicity of multiple lines we construct in this case the perturbed systems corresponding to the given canonical forms associated to the configurations of invariant lines.

Main Theorem D. *Assume that a non-degenerate cubic system (i.e. $\sum_{i=0}^{9} \mu_i^2 \neq 0$) possesses invariant straight lines of total multiplicity 8, including the line at infinity with its own multiplicity. In addition we assume that this system has two distinct infinite singularities, i.e. the conditions $\mathcal{D}_1 = \mathcal{D}_3 = 0$ and $\mathcal{D}_2 \neq 0$ hold. Then:*

I. *This system could not have the infinite singularities defined by two double factors of the invariant polynomial $C_3(x, y)$;*

II. *The system has the infinite singularities defined by one triple and one simple real factors of $C_3(x, y)$ (i.e. $\mathcal{D}_1 = \mathcal{D}_3 = \mathcal{D}_4 = 0$ and $\mathcal{D}_2 \neq 0$) and could possess only one of the 25 possible configurations Config. 8.23 – Config. 8.47 of invariant lines given in Figure 4.1;*

III. *This system possesses the specific configuration Config. 8.j ($j \in \{23, 24, \ldots, 47\}$) if and only if the corresponding conditions included below are fulfilled. Moreover it can be brought via an affine transformation and time rescaling to the canonical form, written below next to the conditions:*

- *Config. 8.23* $\Leftrightarrow N_2N_3 \neq 0, \mathcal{V}_1 = \mathcal{V}_3 = \mathcal{K}_5 = N_1 = N_4 = N_5 = N_6 = N_7 = 0$:
$$\begin{cases} \dot{x} = (x-1)x(1+x), \\ \dot{y} = x - y + x^2 + 3xy; \end{cases}$$

- *Config. 8.24 - 8.27* $\Leftrightarrow N_2 \neq 0,\ N_3 = \mathcal{V}_1 = \mathcal{V}_3 = \mathcal{K}_5 = N_1 = N_4 = N_6 = N_8 = 0, N_9 \neq 0$:
$$\begin{cases} \dot{x} = x(r+2x+x^2), \\ \dot{y} = (r+2x)y,\ r(9r-8) \neq 0; \end{cases} \quad \begin{cases} \textit{Config. 8.24} \Leftrightarrow N_{11} < 0\ (r<0); \\ \textit{Config. 8.25} \Leftrightarrow N_{10} > 0, N_{11} > 0\ (0<r<1); \\ \textit{Config. 8.26} \Leftrightarrow N_{10} = 0\ (r=1); \\ \textit{Config. 8.27} \Leftrightarrow N_{10} < 0\ (r>1); \end{cases}$$

- *Config. 8.28 - 8.30* $\Leftrightarrow N_2 \neq 0,\ N_3 = \mathcal{V}_1 = \mathcal{V}_3 = \mathcal{K}_5 = N_1 = N_5 = N_8 = N_{12} = 0, N_{13} \neq 0$:
$$\begin{cases} \dot{x} = x(r-2x+x^2), (9r-8) \neq 0 \\ \dot{y} = 2y(x-r), r(r-1) \neq 0; \end{cases} \quad \begin{cases} \textit{Config.8.28} \Leftrightarrow N_{15} < 0\ (r<0); \\ \textit{Config.8.29} \Leftrightarrow N_{14} < 0, N_{15} > 0\ (0<r<1); \\ \textit{Config. 8.30} \Leftrightarrow N_{14} > 0\ (r>1); \end{cases}$$

- *Config. 8.31, 8.32* $\Leftrightarrow N_2 = N_3 = \mathcal{V}_1 = \mathcal{V}_3 = \mathcal{K}_5 = N_1 = N_{17} = N_{18} = 0, N_{10}N_{16} \neq 0$:
$$\begin{cases} \dot{x} = x(r+x^2), \\ \dot{y} = x - 2ry,\ r \in \{-1,1\}; \end{cases} \quad \begin{cases} \textit{Config. 8.31} \Leftrightarrow N_{10} < 0\ (r=-1); \\ \textit{Config. 8.33} \Leftrightarrow N_{10} > 0,\ (r=1); \end{cases}$$

- *Config. 8.33* $\Leftrightarrow N_2 = N_3 = 0, \mathcal{V}_1 = \mathcal{V}_3 = \mathcal{K}_5 = N_1 = N_{10} = N_{17} = N_{18} = 0, N_{16} \neq 0$:
$$\begin{cases} \dot{x} = x^3, \\ \dot{y} = 1+x; \end{cases}$$

- *Config. 8.34 - 8.38* $\Leftrightarrow N_2 = N_3 = 0, \mathcal{V}_1 = \mathcal{V}_3 = \mathcal{K}_5 = N_1 = N_{16} = N_{19} = 0, N_{18} \neq 0$:
$$\begin{cases} \dot{x} = x(r+x+x^2), \\ \dot{y} = 1 + ry,\ (9r-2) \neq 0; \end{cases} \quad \begin{cases} \textit{Config. 8.34} \Leftrightarrow N_{21} < 0\ (r<0); \\ \textit{Config. 8.35} \Leftrightarrow N_{20} > 0, N_{21} > 0\ (0<r<1/4); \\ \textit{Config. 8.36} \Leftrightarrow N_{20} = 0\ (r=1/4); \\ \textit{Config. 8.37} \Leftrightarrow N_{20} < 0\ (r>1/4); \\ \textit{Config. 8.38} \Leftrightarrow N_{21} = 0\ (r=0); \end{cases}$$

- *Config.8.39, 8.40* $\Leftrightarrow \mathcal{V}_1 = \mathcal{L}_1 = \mathcal{L}_2 = N_{22} = N_{23} = N_{24} = 0,\ \mathcal{V}_3\mathcal{K}_6 \neq 0$:
$$\begin{cases} \dot{x} = x(r+x+x^2), \\ \dot{y} = (r+2x+3x^2)y; \end{cases} \quad \begin{cases} \textit{Config. 8.39} \Leftrightarrow \mu_6 < 0\ (r<1/4); \\ \textit{Config. 8.40} \Leftrightarrow \mu_6 > 0\ (r>1/4); \end{cases}$$

- *Config.8.41– 8.43* $\Leftrightarrow \mathcal{V}_1 = \mathcal{L}_1 = \mathcal{L}_2 = N_{22} = N_{23} = \mathcal{K}_6 = 0,\ \mathcal{V}_3N_{24} \neq 0$:
$$\begin{cases} \dot{x} = x(r+x^2), \\ \dot{y} = 1 + ry + 3x^2y; \end{cases} \quad \begin{cases} \textit{Config. 8.41} \Leftrightarrow \mu_6 < 0\ (r<0); \\ \textit{Config. 8.42} \Leftrightarrow \mu_6 = 0\ (r=0); \\ \textit{Config. 8.43} \Leftrightarrow \mu_6 > 0\ (r>0); \end{cases}$$

- *Config.8.44–8.47* $\Leftrightarrow \mathcal{V}_5 = \mathcal{U}_2 = \mathcal{K}_4 = \mathcal{K}_5 = \mathcal{K}_6 = N_{24} = N_{25} = N_{26} = N_{27} = 0, \mathcal{V}_1\mathcal{V}_3 \neq 0$:

$$\begin{cases} \dot{x} = x(1+x)[r+2+(r+1)x], \\ \dot{y} = [r+2+(3+2r)x+rx^2]y; \end{cases} \quad \begin{cases} \textit{Config. 8.44} \Leftrightarrow \mu_6 < 0 \ (-2 < r < -1); \\ \textit{Config. 8.45} \Leftrightarrow \mu_6 > 0, N_{28} < 0 \ (r < -2); \\ \textit{Config. 8.46} \Leftrightarrow \mu_6 > 0, N_{28} > 0 \ (r > -1); \\ \textit{Config. 8.47} \Leftrightarrow \mu_6 = 0 \ (r = -1). \end{cases}$$

The proof of Main Theorem D helped us to detect a new class of systems in $\mathbb{CSL}_9$ completing the classification given by Llibre and Vulpe in [25]. And namely, we arrive at the system (S_9): $\dot{x} = x(2+3x)(4+3x)/9$, $\dot{y} = 2(4+9x)y/9$ which possesses invariant lines of total multiplicity 9: $x = 0$ (triple), $x = -2/3$ (double), $x = -4/3$ and $y = 0$ (both simple) and the line at infinity: $Z = 0$ (double).

Considering the configuration of invariant lines of this system given in Figure 4.16 we observe that this configuration is different from configurations given in Figures 14–22 [25].

GENERAL CONCLUSIONS AND RECOMMENDATIONS

The Thesis is devoted to the problem of classifying a family of cubic systems which possess invariant straight lines according to the configurations of these lines. This problem in qualitative study of differential equations, which is very hard even in the simplest case of quadratic differential systems, is partly motivated by *the problem of topologically classifying all phase portraits* of polynomial cubic systems.

As a general observation we note that cubic differential systems are harder to study than quadratic differential systems because of phenomena which could occur in this class but which never occur in the quadratic family. For example the joint presence of limit cycles and singularities which are centers is a phenomenon which occurs in cubic differential systems but does not occur in the quadratic family. Furthermore cubic systems form a family depending on 20 parameters while the class of quadratic differential systems depends on only 12 parameters.

Here we consider the family $\mathbb{CSL}_8$ of cubic systems with invariant lines of total multiplicity eight (including the line at infinity). The study of these systems was based on some concepts, such as the invariant straight line, the multiplicity of lines (of finite/infinite singular points) and the configuration of invariant lines.

An other aspect of the practical and theoretical values of the work is that having all canonical forms of systems in $\mathbb{CSL}_8$, constructed in the current Thesis, *the problem of integrability* of such systems could be solved. Of course we realize that, at the first glance,

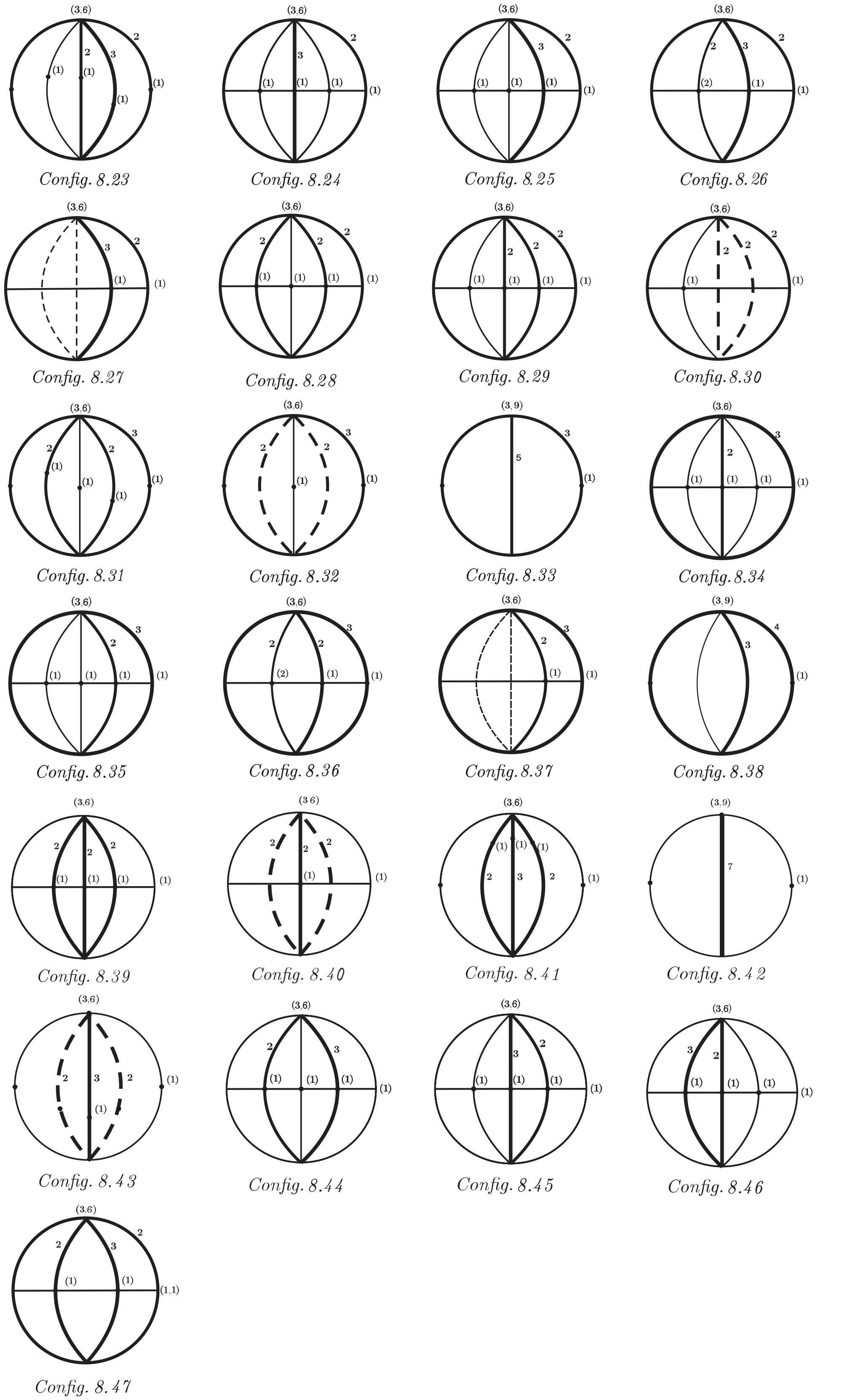


Fig. 4.1. Configurations of invariant lines for systems in $\mathbb{CSL}_8$ with 2 $\mathbb{ISP}s$

the class $\mathbb{CSL}_8$ is a very specific one, moreover the cases of integrable systems are rare, but as Arnold said in [1, p.405] "...these integrable cases allow us to collect a large amount of

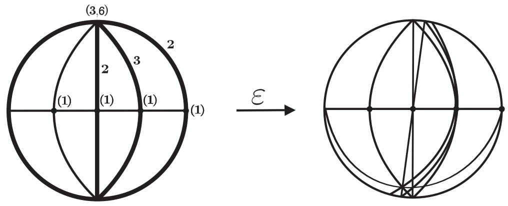


Fig. 4.16. Configuration of $\mathbb{IL}s$ corresponding to system (S_9)

information about the motion in more important systems...“.

The main scientific problem which is solved in this Thesis consists in classifying the whole family of cubic differential systems possessing invariant lines of total multiplicity eight according to configurations of these lines; this classification is very helpful for obtaining the complete topological classification of this family and is useful for the study of integrability of this systems.

Novelty and scientific originality of the work consists in the fact that for the first time there are constructed all the possible configurations of invariant lines for systems in $\mathbb{CSL}_8$ and the obtained results are reflected in [8-26]. This set of configurations contains as particular cases all the configurations detected by other authors in special cases of systems in $\mathbb{CSL}_8$ (see [26], [29, 30]). In fact, this work is a continuation of [25] where the cubic systems with the maximum number of invariant lines (i.e. 9) were studied and where 23 configurations of invariant lines are detected. However a new class of cubic systems possessing invariant lines of total multiplicity nine which completes the classification given by Llibre and Vulpe in [25] was detected in this Thesis. Thus, we have obtained 51 different configurations of invariant lines. More exactly, we have detected 17 (respectively 5; 25; 4) distinct configurations for the subfamily of systems possessing four (respectively three; two; one) distinct infinite singularities, real or/and complex. At the same time we have constructed 33 canonical forms of the systems possessing invariant lines of total multiplicity eight. We point out that 19 of these canonical forms are one-parameter families, whereas 14 of them are systems with constant coefficients. One more result obtained in the Thesis is the construction of perturbed canonical forms which prove that taking jointly the invariant lines they produced a maximum of eight distinct invariant lines.

The benefits of our elaborations are that this classification, which is taken modulo the action of the group of the real affine transformations and time rescaling, is given in terms of invariant polynomials. Therefore the author of the current Thesis has constructed 52 new invariant polynomials besides 20, which were constructed in [25]. It is worth to mention that it was made a great work, because the process of the construction of invariant polynomials takes time and it is pretty difficult. These algebraic invariants and comitants allow one to verify for any given real non-degenerate cubic system with non-degenerate infinity whether or not it has invariant lines of total multiplicity eight, and to specify its configuration of lines endowed with their corresponding real singularities of this system. The important fact is that the calculations can be implemented on computer.

In addition to complex investigations on the research problem, the contribution of the author is materialized by the following **main conclusions** of the Thesis:

1) In this Thesis we studied a whole family of cubic systems, i.e those possessing invariant straight lines of total multiplicity 8. We show that for this class, which we denote by $\mathbb{CSL}_8$, several normal forms are needed each one depending of at most one parameter. To obtain *global results* we used the invariant theory of polynomial differential systems as developed by Sibirschi and his school. *This method allowed us to glue in a unique global diagram, the bifurcation diagrams of the several normal forms needed in the study of this family.*

2) The global result mentioned above is a bifurcation diagram in the 20 dimensional parameter space of cubic differential systems. *This gives us an algorithm to decide for each cubic system whether it belongs to this family or not and in case it belongs to this family, it allows us to effectively compute its specific configuration of invariant lines. We proved that this family possesses **51** possible configurations of invariant straight lines.*

3) The Thesis also led us to obtain the following *new global results*:

(i) a system in $\mathbb{CSL}_8$ must have at infinity at least one real singularity;

(ii) a cubic system with real infinite singularities defined by two double factors of the invariant polynomial $C_3(x, y) = yp_3(x, y) - xQ_3(x, y)$ could not belong to the class $\mathbb{CSL}_8$ and this is the unique exception of cubic systems with real infinite singularities.

We propose the following **recommendations:** (a) the configurations of invariant lines detected, and canonical forms could be used for a complete topological classification of cubic systems in this class; (b) the canonical forms of cubic systems in $\mathbb{CSL}_8$ constructed can serve as a basis for determining of the first integrals of such systems; (c) to use the polynomial invariant we constructed for further investigations of cubic systems with invariant lines of total multiplicity less than 8; (d) to apply the scientific results obtained, in the study of some mathematical models which are described by polynomial differential systems and which are related with some problems in physics, chemistry, medicine, etc. (e) these investigations could serve as a support for teaching courses in higher education.

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C.Z.U: 517.925

ADNOTARE

Bujac Cristina, "Sisteme diferenţiale cubice cu drepte invariante de multiplicitate totală opt", doctor în ştiinţe matematice, Chişinău, 2016.

Lucrarea este scrisă în limba engleză, conţine 154 pagini text de bază şi are următoarea structură: introducere, 4 capitole, concluzii generale şi recomandări, bibliografia (care include 140 titluri). Rezultatele obţinute sunt publicate în 19 lucrări ştiinţifice.

Cuvintele cheie: sistem differenţial cubic, polinom afin invariant, dreaptă invariantă, multiplicitatea curbei algebrice, configuraţie de drepte invariante, sistem perturbat.

Domeniul de studiu al tezei: teoria calitativă a sistemelor dinamice, teoria invarianţilor algebrici a ecuaţiilor diferenţiale.

Scopul şi obiectivele lucrării: de a efectua clasificarea completă a familiei de sisteme cubice cu drepte invariante de multiplicitate totală 8; această clasificare presupune determinarea tuturor configuraţiilor de drepte invariante posibile pentru această familie de sisteme cubice şi construirea condiţiilor necesare şi suficiente afin invariante pentru realizarea fiecarei dintre configuraţiile depistate.

Noutatea şi originalitatea ştiinţifică. În lucrare au fost construite pentru prima dată toate configuraţiile posibile de drepte invariante de multiplicitate totală opt ale familiei de sisteme diferenţiale cubice. Această mulţime de configuraţii conţine toate configuraţiile depistate de alţi autori pentru unele clase speciale de sisteme cubice. Adiţional, s-au determinat condiţiile necesare şi suficiente afin-invariante pentru realizarea fiecăreia dintre configuraţiile construite. De asemenea a fost completată clasificarea realizată de Llibre şi Vulpe depistând o noua clasă de sisteme cubice cu drepte invariante de multiplicitate totală nouă.

Problema ştiinţifică importantă soluţionată constă în clasificarea completă a familiei de sisteme cubice cu drepte invariante de multiplicitate totală opt în raport cu configurţtiile acestor drepte; aceasta clasificare este un element foarte util în vederea clasificării topologice complete ale acestei familii de sisteme şi în vederea studiului integrabilităţii acestor sisteme.

Semnificaţia teoretică şi valoarea aplicativă a lucrării. Rezultatele ce ţin de sistemele cubice cu drepte invariante de multiplicitate totală opt obţinute în teză reprezintă un pas important în studiul algebro-geometric al familiei de sisteme cubice diferenţiale bidimensionale.

Implementarea rezultatelor ştiinţifice: *(i)* drept bază pentru determinarea integralelor prime ale acestor sisteme; *(ii)* pentru investigarea ulterioară a sistemelor cubice cu drepte invariante de multiplicitate mai mică decât 8; *(iii)* în studiul diverselor modele matematice care descriu diferite procese din fizică , chimie, medicină ş.a.m.d.; *(iv)* în calitate de suport pentru perfectarea cursurilor speciale universitare şi post-universitare.

УДК 517.925

АННОТАЦИЯ

Bujac Cristina, “ Кубические дифференциальные системы с инвариантными прямыми суммарной кратности восемь”, степень доктора математических наук, Chişinău, 2016.

Работа написана на английском языке. Она состоит из введения, 4-х глав, общих выводов и рекомендаций, 140 источников литературы, 154 страниц основного текста. Полученные результаты опубликованы в 19 научных работах.

Ключевые слова: кубическая дифференциальная система, аффинно-инвариантный полином, инвариантная прямая, кратность прямой, конфигурация инвариантных прямых, возмущенные системы.

Цель и задачи диссертации: построить полную классификацию семейства плоских кубических систем дифференциальных уравнений в соответствии с конфигурациями инвариантных прямых общей кратности восемь, а именно: определить все возможные такие конфигурации и построить необходимые и достаточные аффинно-инвариантные условия для реализации каждого из обнаруженных конфигураций.

Область исследования: Качественная теория динамических систем, теория инвариантов дифференциальных уравнений.

Научная новизна и оригинальность. В диссертации впервые построены все возможные конфигурации инвариантных прямых суммарной кратности восемь для семьи плоских кубических систем дифференциальных уравнений. Этот набор конфигураций содержит все конфигурации, другими авторами для частных классов кубических систем. Кроме того, мы определили необходимые и достаточные условия для реализации каждой из полученных конфигураций. Дополнительно обнаружили новый класс кубических систем с инвариантными прямыми суммарной кратности 9, тем самым дополняя классификацию Llibre и Vulpe.

Основная решенная научная задача состоит в полной классификации двумерных кубических систем дифференциальных уравнений в соответствии с их конфигурациями инвариантных прямых общей кратности 8, основанной на применении теории инвариантов дифференциальных уравнений. Эта классификация генерирует полезную базу для дальнейшей полной топологической классификации данного семейства систем.

Теоретическое и практическое значение работы. Полученные в данной работе результаты, касающиеся кубических систем с инвариантными прямыми суммарной кратностью 8, представляют собой важный шаг в классификации всего множества кубических систем.

Реализация научных результатов. Результаты могут быть применены: *(i)* в качестве основы для определения первых интегралов таких систем; *(ii)* для дальнейших исследований более общих кубических систем с инвариантными прямыми суммарной кратностью менее чем 8; *(iii)* в изучении некоторых математических моделей, описывающих процессы в физике, химии, медицине и т.д.; *(iv)* для разработки специальных курсов в системе высшего образования.

C.Z.U: 517.925

ANNOTATION

Bujac Cristina, " Cubic differential systems with invariant lines of total multiplicity eight ", Doctor degree in Mathematics, Chişinău, 2016.

The language of the Thesis is English. It comprises 154 base pages and has the following structure: Introduction, 4 Chapters, General Conclusions and Recommendations, Bibliography with 140 References. Research outcomes were reflected in 19 scientific publications.

Keywords: cubic differential system, affine invariant polynomial, invariant straight line, multiplicity of a line, configuration of invariant straight lines, perturbed system.

Field of study: Qualitative Theory of Dynamical Systems, Invariant Theory of Differential Equations.

The purpose and objectives: to give a full classification for the family of cubic systems with invariant straight lines of total multiplicity eight; this classifications supposes the detection of all possible configurations of invariant lines for this family and the construction of affine invariant criteria for the realization of each one of the detected configurations.

Novelty and scientific originality. In our Thesis for the first time there are constructed all the possible configurations of invariant lines of total multiplicity eight for cubic systems. Our set of configurations contains as particular cases all the configurations detected by other authors in special cases. Additionally we give necessary and sufficient conditions for the realization of each one of the corresponding configurations. Moreover we completed the classification of Llibre and Vulpe detecting a new class of cubic systems with invariant lines of total multiplicity nine.

The main scientific problem which is solved in this Thesis consists in classifying the whole family of cubic differential systems possessing invariant lines of total multiplicity eight according to configurations of these lines; this classification is very helpful for obtaining the complete topological classification of this family and is useful for the study of integrability of these systems.

The significance of theoretical and practical values of the work. The obtained in this thesis results concerning cubic systems with invariant lines of total multiplicity eight represent an important step in algebraic and geometric studies of cubic differential systems.

Implementation of the scientific results. They could be applied: *(i)* as a basis for determining of the first integrals of such systems; *(ii)* for further investigations of cubic systems with invariant lines of total multiplicity less then 8; *(iii)* in the study of some mathematical models which describe processes in physics, chemistry, medicine and so on; *(iv)* as a support for teaching courses in higher education.

BUJAC CRISTINA

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111.02 - Ecuaţii Diferenţiale

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