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CUBIC DIFFERENTIAL SYSTEMS WITH INVARIANT LINES OF TOTAL MULTIPLICITY EIGHT

111.02 - Differential Equations

Doctor Thesis in Mathematics

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SISTEME DIFERENŢIALE CUBICE CU DREPTE INVARIANTE DE MULTIPLICITATE TOTALĂ OPT

111.02 - Ecuaţii Diferenţiale

Teza de doctor în ştiinţe matematice

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ADNOTARE

Bujac Cristina, "Sisteme diferenţiale cubice cu drepte invariante de multiplicitate totală opt", doctor în ştiinţe matematice, Chişinău, 2016.

Lucrarea este scrisă în limba engleză, conţine 154 pagini text de bază şi are următoarea structură: introducere, 4 capitole, concluzii generale şi recomandări, bibliografia (care include 140 titluri). Rezultatele obţinute sunt publicate în 19 lucrări ştiinţifice.

Cuvintele cheie: sistem differenţial cubic, polinom afin invariant, dreaptă invariantă, multiplicitatea curbei algebrice, configuraţie de drepte invariante, sistem perturbat.

Domeniul de studiu al tezei: teoria calitativă a sistemelor dinamice, teoria invarianţilor algebrici a ecuaţiilor diferenţiale.

Scopul şi obiectivele lucrării: de a efectua clasificarea completă a familiei de sisteme cubice cu drepte invariante de multiplicitate totală 8; această clasificare presupune determinarea tuturor configuraţiilor de drepte invariante posibile pentru această familie de sisteme cubice şi construirea condiţiilor necesare şi suficiente afin invariante pentru realizarea fiecarei dintre configuraţiile depistate.

Noutatea şi originalitatea ştiinţifică. În lucrare au fost construite pentru prima dată toate configuraţiile posibile de drepte invariante de multiplicitate totală opt ale familiei de sisteme diferenţiale cubice. Această mulţime de configuraţii conţine, în calitate de cazuri particulare, toate configuraţiile depistate de alţi autori pentru unele clase speciale de sisteme cubice. Adiţional, s-au determinat condiţiile necesare şi suficiente afin-invariante de realizare ale configuraţiilor construite. De asemenea a fost completată clasificarea realizată de Llibre şi Vulpe depistând o noua clasă de sisteme cubice cu drepte invariante de multiplicitate totală nouă.

Problema ştiinţifică importantă soluţionată constă în clasificarea completă a familiei de sisteme cubice cu drepte invariante de multiplicitate totală opt în raport cu configurąţiile acestor drepte; aceasta clasificare este un element foarte util în vederea clasificării topologice complete ale acestei familii de sisteme şi în vederea studiului integrabilităţii acestor sisteme.

Semnificaţia teoretică şi valoarea aplicativă a lucrării. Rezultatele ce ţin de sistemele cubice cu drepte invariante de multiplicitate totală opt obţinute în teză reprezintă un pas important în studiul algebro-geometric al familiei de sisteme cubice diferenţiale bidimensionale.

Implementarea rezultatelor ştiinţifice: *(i)* drept bază pentru determinarea integralelor prime ale acestor sisteme; *(ii)* pentru investigarea ulterioară a sistemelor cubice cu drepte invariante de multiplicitate mai mică decât 8; *(iii)* în studiul diverselor modele matematice care descriu diferite procese din fizică , chimie, medicină ş.a.m.d.; *(iv)* în calitate de suport pentru perfectarea cursurilor speciale universitare şi post-universitare.

УДК 517.925

АННОТАЦИЯ

Bujac Cristina, " Кубические дифференциальные системы с инвариантными прямыми суммарной кратности восемь", степень доктора математических наук, Chişinău, 2016.

Работа написана на английском языке. Она состоит из введения, 4-х глав, общих выводов и рекомендаций, 140 источников литературы, 154 страниц основного текста. Полученные результаты опубликованы в 19 научных работах.

Ключевые слова: кубическая дифференциальная система, аффинно-инвариантный полином, инвариантная прямая, кратность прямой, конфигурация инвариантных прямых, возмущенные системы.

Цель и задачи диссертации: построить полную классификацию семейства плоских кубических систем дифференциальных уравнений в соответствии с конфигурациями инвариантных прямых общей кратности восемь, а именно: определить все возможные такие конфигурации и построить необходимые и достаточные аффинно-инвариантные условия для реализации каждого из обнаруженных конфигураций.

Область исследования: Качественная теория динамических систем, теория инвариантов дифференциальных уравнений.

Научная новизна и оригинальность. В диссертации впервые построены все возможные конфигурации инвариантных прямых суммарной кратности восемь для семьи плоских кубических систем дифференциальных уравнений. Этот набор конфигураций содержит все конфигурации, определенные другими авторами для частных классов кубических систем. Кроме того, мы определили необходимые и достаточные условия для реализации каждой из полученных конфигураций. Дополнительно обнаружили новый класс кубических систем с инвариантными прямыми суммарной кратности 9, тем самым дополняя классификацию Llibre и Vulpe.

Основная решенная научная задача состоит в полной классификации двумерных кубических систем дифференциальных уравнений в соответствии с их конфигурациями инвариантных прямых общей кратности 8, основанной на применении теории инвариантов дифференциальных уравнений. Эта классификация генерирует полезную базу для дальнейшей полной топологической классификации данного семейства систем.

Теоретическое и практическое значение работы. Полученные в данной работе результаты, касающиеся кубических систем с инвариантными прямыми суммарной кратностью 8, представляют собой важный шаг в классификации всего множества кубических систем.

Реализация научных результатов. Результаты могут быть применены: *(i)* в качестве основы для определения первых интегралов таких систем; *(ii)* для дальнейших исследований более общих кубических систем с инвариантными прямыми суммарной кратностью менее чем 8; *(iii)* в изучении некоторых математических моделей, описывающих процессы в физике, химии, медицине и т.д.; *(iv)* для разработки специальных курсов в системе высшего образования.

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ANNOTATION

Bujac Cristina, " Cubic differential systems with invariant lines of total multiplicity eight ", Doctor degree in Mathematics, Chişinău, 2016.

The language of the Thesis is English. It comprises 154 base pages and has the following structure: Introduction, 4 Chapters, General Conclusions and Recommendations, Bibliography with 140 References. Research outcomes were reflected in 19 scientific publications.

Keywords: cubic differential system, affine invariant polynomial, invariant straight line, multiplicity of a line, configuration of invariant straight lines, perturbed system.

Field of study: Qualitative Theory of Dynamical Systems, Invariant Theory of Differential Equations.

The purpose and objectives: to give a full classification for the family of cubic systems with invariant straight lines of total multiplicity eight; this classifications supposes the detection of all possible configurations of invariant lines for this family and the construction of affine invariant criteria for the realization of each one of the detected configurations.

Novelty and scientific originality. In our Thesis for the first time there are constructed all the possible configurations of invariant lines of total multiplicity eight for cubic systems. Our set of configurations contains as particular cases all the configurations detected by other authors in special cases. Additionally we give necessary and sufficient conditions for the realization of each one of the corresponding configurations. Moreover we completed the classification of Llibre and Vulpe detecting a new class of cubic systems with invariant lines of total multiplicity nine.

The main scientific problem which is solved in this Thesis consists in classifying the whole family of cubic differential systems possessing invariant lines of total multiplicity eight according to configurations of these lines; this classification is very helpful for obtaining the complete topological classification of this family and is useful for the study of integrability of these systems.

The significance of theoretical and practical values of the work. The obtained in this thesis results concerning cubic systems with invariant lines of total multiplicity eight represent an important step in algebraic and geometric studies of cubic differential systems.

Implementation of the scientific results. They could be applied: *(i)* as a basis for determining of the first integrals of such systems; *(ii)* for further investigations of cubic systems with invariant lines of total multiplicity less then 8; *(iii)* in the study of some mathematical models which describe processes in physics, chemistry, medicine and so on; *(iv)* as a support for teaching courses in higher education.

INTRODUCTION

Actual research status and importance of investigated problem. The polynomial differential systems are objects of numerous scientific investigations. These systems occur in many branches of applied mathematics and they also have theoretical importance. A major driving force in the development of their theory was a collection of very difficult problems some of which are over one hundred years old. A source of these difficulties is nonlinearity of mathematical models of complicated real processes and devices. These problems also motivated the work we present in this paper.

The study of planar polynomial vector fields began to be pursued after the publication of the famous papers of Darboux [59] (1878) and of Poincaré [89] (1881), [91] (1885), [92] (1891). Darboux'work was on integrability in terms of existence of algebraic invariant curves of complex polynomial differential equations over the complex projective plane (a modern presentation of the theory of Darboux can be found in [42, 107]). This very beautiful work which could also be applied to compute first integrals of real polynomial differential equations, was very much admired by Poincaré and motivated his statement in 1891 [92] of a very hard problem still open today, on planar polynomial differential systems (see further below). The main motivation of Poincaré for studying these systems came from his interest in the problem of the stability of the solar system. This problem asks if in the very long term the solar system will preserve its present state or whether major changes such as a planet escaping from the system or a collision among bodies in the system will occur. As this problem is very hard, Poincaré decided to begin by first studying the simplest non-linear differential equations which are the planar polynomial ones and he wrote the two seminal papers [89], [91] which founded *the qualitative study of differential equations.* More precisely, even if the differential equation can not be solved in terms of known functions, yet from the very form of the equation, a wealth of information about the geometric properties and behavior of the solutions can be found. Rather than emphasizing calculations of specific solutions, Poincaré took the global approach by considering the solutions in their totality. In these works he introduced many new notions, for example for special types of singularities such as foci, nodes, saddles, centers which distinguished the behavior of solutions around singularities; the notion of limit cycle - an isolated periodic solution in the set of all periodic solutions of a system; the notion of the Poincaré first return map, etc. In these papers Poincaré proved a number of theorems, among them the theorem saying that a necessary and sufficient condition for a polynomial vector field on the plane to have a center at a singular point with purely imaginary eigenvalues, is that the system admit a non-zero local analytic first integral in a neighborhood of this singular point. In his Mémoire *The general problem of the stability of motion* [79], Liapunov extended this theorem for analytic differential systems (actually Liapunov studied differential systems in n variables but when results are applied to the case $n = 2$ systems we obtain this more general theorem for analytic two-dimensional systems). In this mémoire Liapunov developed the theory of stability of motion.

Poincaré stated two problems on polynomial differential equations on the plane: *the problem of the center* [91] and *the problem of algebraic integrability* [92] of such equations. One way in which we can state the problem of the center is the following: given a positive integer n find the necessary and sufficient conditions for a polynomial system of degree n to have a singularity which is a center. In fact Poincaré only considered singularities with a non-degenerate linear part of focus or center type. More precisely he considered the case where the eigenvalues are purely imaginary $\pm\beta i$, $\beta \in \mathbb{R}\backslash\{0\}$ and in this case the problem of the center is to give conditions for distinguishing between a center and a focus. The problem of algebraic integrability was stated by Poincaré in 1891 in [92]. This problem asks for necessary and sufficient conditions for a polynomial differential system to have a rational first integral. Both problems are of a global nature involving whole classes of polynomial differential systems and this is one of the reasons they are so hard. Apart from the quadratic case for which only one of these two problems, the problem of the center, was solved, both problems remain open for any natural number $n \geq 3$.

A third famous problem on planar polynomial systems is *the second part of Hilbert's 16th problem* stated in the list of 23 problems posed by Hilbert in his address at the International Mathematical Congress in Paris in 1900. This problem asks to determine for any natural number n the maximum number of limit cycles which a planar polynomial differential system of degree n could have and it remains to be one of the most difficult problems to be solved. The interest is in the global behavior of all solutions in the whole plane and even at infinity (cf. [66]) and this for a whole family of systems, which is why this problem is so hard.

In [93] Poincaré posed the *individual finiteness problem* which asks to prove that any individual polynomial differential system has a finite number of limit cycles. Poincaré solved this problem in a special case by proving the individual finiteness theorem for systems which could only have simple graphics.

An even harder problem than Hilbert's 16th problem for polynomial differential systems is *the problem of topologically classifying all phase portraits of polynomial systems of a given degree n*. This problem is very hard even in the simplest case of quadratic differential systems. There are several subclasses of the quadratic and cubic classes for which this problem was solved (we refer the reader to [117, 127, 128]).

Our work in Theses was partly motivated by the problem of topologically classifying the cubic differential systems. We are interested in the investigation of the polynomial differential cubic systems with invariant straight lines. Here we pose the problem of its classification according to the configurations of invariant straight lines, which generates an useful base for a further total topological classification of this family of systems.

The existence of sufficiently many invariant straight lines of planar polynomial systems could be used for integrability of such systems. During the past 15 years several articles were published on this theme. Investigations concerning polynomial differential systems possessing invariant straight lines were done by Popa, Sibirski, Kooij, Sokulski, Zhang Xi Kang, Schlomiuk, Vulpe, Dai Guo Ren, Artes, Llibre as well as Dolov and Kruglov.

The set $\mathbb{CS}$ of cubic differential systems depends on 20 parameters and for this reason people began by studying particular subclasses of $\mathbb{CS}$. We mention here some references on polynomial differential systems possessing invariant straight lines. For quadratic systems see [62, 101, 102, 108, 110–112] and [114]; for cubic systems see [16, 17, 27, 28, 78, 83, 85, 86, 103, 127] and [128]; for quartic systems see [123] and [139]; for some more general systems see [74, 96, 97] and [100].

According to [5] the maximum number of invariant straight lines taking into account their multiplicities for a polynomial differential system of degree m is $3m$ when we also consider the infinite straight line. This bound is always reached if we consider the real and the complex invariant straight lines, see [43].

So the maximum number of the invariant straight lines (including the line at infinity $Z = 0$) for cubic systems with non-degenerate infinity is 9. A classification of all cubic systems possessing the maximum number of invariant straight lines taking into account their multiplicities has been made in [83]. The authors used the notion of *configuration of invariant lines* for cubic systems (as introduced in [114]) and detected 23 such configurations. Moreover using invariant polynomials with respect to the action of *the group* $Aff(2, \mathbb{R})$ *of affine transformations and time rescaling* in this paper, the necessary and sufficient conditions for the realization of each one of 23 configurations were detected. A new class of cubic systems omitted in [83] was constructed in [16].

This paper is a continuation of [83]. More exactly, here we shall consider the family of cubic systems with invariant lines of total multiplicity eight, including the line at infinity and considering their multiplicities (we denote this family by $\mathbb{CSL}_8$). The results concerning these systems are exhibit in [12–30]

Some systems in $\mathbb{CSL}_8$ have been also investigated by Lyubimova [86], Şubă, Puţuntică and Repeşco [127, 128]. Lyubimova considered such cubic systems with invariant lines, all real and distinct, and constructed 3 configurations of invariant straight lines (and 4 phase portraits). Şubă and his coauthors using the notion of *parallel multiplicity* arrived at 17 configurations of invariant lines which coincide with those obtained in our classification in the case of cubic systems with four distinct infinite singularities. But in contrast with their work, for each configuration we give the necessary and sufficient conditions for its realization in terms of invariant polynomials with respect to the group of affine transformations and time rescaling. We note that the invariant polynomials was constructed applying the Invariant Theory of Differential Equations, founded by C. Sibirschi and developed by his disciples (Lunchevici, Marinciuc, Gasinschi–Chirniţchi, Dang Dini Bic, Tacu, Vulpe, Popa, Boularas Driss, Baltag, Calin, Daniliuc, etc.).

The invariant theory is one of the important tools used in the qualitative study of polynomial differential systems. This theory allows to characterize geometric properties of a given differential systems which remain invariant under the action of a given group of transformations, with the help of algebraic or semi-algebraic relations depending on the coefficients of these systems. Thus the theory of invariant is proven useful in the qualitative studies of polynomial differential systems, in particular to establish invariant (necessary and/or sufficient) conditions in relation to the given group of transformations, that give the existence and the nature of singular points, characterize normal forms or the number of complete lines, give the existence of parallel invariant straight lines, and so on. The computation of invariants, however still difficult. Indeed, for planar cubic differential systems, the invariants are polynomials of 20 indeterminate. The Computer algebra become an indispensable mean when using the theory of invariants. Indeed, the qualitative study of polynomial differential systems leads on algebraic systems. The spectacular progress of the computer algebra and the efficient of the software (Maple, Wolfram Mathematica, P4, etc.) motivate our work.

As it was mentioned above that the main object of our investigations are cubic systems possessing invariant lines of total multiplicity eight. Of course we realize that, at the first glance, the class $\mathbb{CSL}_8$ of cubic systems is a very specific one and we have to point out that this class is even a class of integrable systems (see for instance [128] where this fact is proved for a subclass of $\mathbb{CSL}_8$). The cases of integrable systems are rare, but as Arnold said in [1, p.405] "*...these integrable cases allow us to collect a large amount of information about the motion in more important systems...*".

The purpose and objectives of the thesis. The main goal of the Thesis is to give a full classification of cubic systems with invariant straight lines of total multiplicity eight. This classification involves the realization of the following objectives:

1. to detect all possible configurations of invariant straight lines for this family of systems;

2. to construct necessary and sufficient affine invariant conditions for the realization of each one of the detected configurations.

Novelty and scientific originality. In our Thesis for the first time there are constructed all possible configurations of invariant lines of total multiplicity eight for cubic systems. Our set of configurations contains as particular cases all the configurations detected by other authors for special cases of systems in $\mathbb{CSL}_8$ (see [86], [127, 128]). But in contrast with these papers in the Thesis we have constructed necessary and sufficient conditions for the realization of each one of the corresponding configurations. Moreover we detect a new class of cubic systems with invariant lines of total multiplicity nine.

Methodology of scientific study. The reaserch carried out in the current Thesis is based on methods of Qualitative Theory of Dynamical Systems, Invariant Theory of Differential Equations, methods of Bifurcation Theory of Dynamic Systems, methods of Algebraic Computations.

The main scientific problem which is solved in this Thesis consists in classifying the whole family of cubic differential systems possessing invariant lines of total multiplicity eight according to configurations of these lines; this classification is very helpful for obtaining the complete topological classification of this family and is useful for the study of integrability of these systems.

Principal scientific results to be defended:

(a) all possible 51 configurations of invariant straight lines for cubic systems possessing invariant lines of total multiplicity eight;

(b) the necessary and sufficient affine invariant conditions for the realization of each one of 51 configurations;

(c) the representatives of the family of systems with invariant lines of total multiplicity eight modulo the action of the affine group and time rescaling;

(d) the perturbed canonical systems which characterize the vicinities of cubic systems in $\mathbb{CSL}_8$;

(e) a new class of cubic systems possessing invariant lines of total multiplicity nine which completes the classification given by Llibre and Vulpe in [83].

Implementation of the scientific results. The scientific results obtained could be used for a deeper investigation of cubic systems possessing invariant straight lines of total multiplicity eight (including the line at infinity), and namely:

- the configurations of invariant lines detected, and canonical forms could be used for a complete topological classification of cubic systems in this class;

- the canonical forms constructed for cubic systems in $\mathbb{CSL}_8$ can serve as a basis for determining of the first integrals of such systems;

- the necessary and sufficient affine invariant conditions can be applied for any cubic system in order to detect if it belongs to $\mathbb{CSL}_8$ and if so, then to specify its configuration of invariant lines;

- this classification could be helpful for further investigations of cubic systems with invariant lines of total multiplicity less than 8;

- scientific results obtained can be applied in the study of some mathematical models which are described by polynomial differential systems and which are related with some problems in physics, chemistry, medicine and so on.

- these investigations could serve as a support for teaching courses in higher education.

Approval of obtained scientific results. The scientific results obtained and to be defined were examined and approved by various research seminars, which are as follows: Qualitative Theory of Differential Equations of Moldova State University, 2015; Differential Equations and Algebras of Tiraspol State University, 2013, 2014; seminar of the Department of Differential Equations and Systems Analysis of Belorussian State University, Minsk, 2013; seminar of the Department of Mathematics of the Shanghai Normal University, Shanghai (China), 2015.

Main scientific results included in the Thesis were presented at several scientific conferences: International Conference of Young Researchers, X-th Edition, Chişinău, 2012; Conference on Applied and Industrial Mathematics (CAIM), Chişinău: U.S.T., 2012, 2014, 2015; International Conference “Mathematics and Information Technologies: Research and Education" (MITRE), Chişinău: U.S.M., 2013-2015; Conferinţa Ştiinţifică Internaţională a doctoranzilor "Tendinţe Contemporane ale Dezvoltării Ştiinţei: Viziuni ale Tinerilor Cercetători", Chişinău: AŞM, 2014, 2015; The Third Conference of Mathematical Society of Moldova (IMCS-50), Chişinău: AŞM, 2014; Conferinţa Ştiinţifică Internaţională cu participare internaţională "Probleme actuale ale ştiinţelor exacte şi ale naturii", Chişinău: U.S.T., 2015.

Research papers. Research outcomes are reflected in 19 publications: 3 preprints, 6 scientific peer-reviewed articles, 10 proceedings and abstracts of international conferences; 2 articles and 6 abstracts are published as single-author papers.

Keywords: cubic differential system, group of affine transformations, invariant polynomial, invariant straight line, multiplicity of a line, configuration of invariant straight lines, type of configuration, canonical form, perturbed system.

The thesis is devoted to the research in the following scientific field: Qualitative Theory of dynamical systems, Invariant Theory of differential equations.

Structure of the Thesis. The Thesis is written in English on 154 base pages and has the following structure: Introduction, 4 Chapters, General Conclusions and Recommendations, Bibliography with 140 References. Additionally the Thesis includes 28 figures.

The **Introduction** reveals the actual status of the conducted research, main reasons for carrying on the proposed research, the purpose and objectives of the thesis, the importance and advantages of the conducted scientific investigations, novelty and scientific originality, scientific and research problems solved, the scientific results to be defended, as well as the approval of obtained scientific results.

Chapter 1 contains a survey of the most important results related to the purpose and objectives of the Thesis. In the fist section we give a brief survey on cubic differential systems with invariant straight lines. More exactly, we discuss about the qualitative theory of differential systems and the importance to study the configurations of invariant lines for cubic systems which serve as a a basis for completing the phase portraits of the corresponding systems. So, our Thesis was partly motivated by the problem of topologically classifying the cubic differential systems. In the second section we describe the problem of integrability concerning planar differential systems. Having obtained all canonical forms for cubic systems possessing invariant lines of total multiplicity eight, the problem of integrability of such systems could be resolved and this also motivated our work. The last section is devoted to the concept of invariant polynomial and its use in classification problems. We briefly review the classical theory of invariants and its analog for the theory of polynomial vector fields developed by Sibirskii school and its new developments by the joint work of the Chişinău school, the Barcelona school and by Schlomiuk.

In **Chapter 2** In Chapter 2 we firstly give the preliminary definitions and results needed in the work. This section is devoted to some aspects concerning the Invariant Theory and besides some invariant polynomials earlier constructed we exhibit 52 new invariant polynomials, which are in fact $CT-$comitants. We also describe the scheme of the proofs of the main theorems. In Paragraph 2.2 we state and prove the classification theorem (Main Theorem A) of cubic systems in $\mathbb{CSL}_8$ having four distinct infinite singular points (denoted by $\mathbb{ISP}s$) according to configurations of invariant lines and for each configuration we give the corresponding necessary and sufficient conditions for its realization in terms of algebraic invariants and comitants with respect to the group of affine transformations and time rescaling. For the family of such cubic systems we also construct their representatives modulo action of the group under consideration.

Chapter 3 is devoted to the proofs of two classification theorems: Main Theorem B which deals with cubic systems in $\mathbb{CSL}_8$ possessing three distinct $\mathbb{ISP}s$ and Main Theorem C which is related to cubic systems in $\mathbb{CSL}_8$ possessing exactly one infinite singularity. So we detect all possible configurations of $\mathbb{ISL}s$ for these two subfamilies of systems and for each configuration we give the necessary and sufficient conditions for its realization. In order to prove the multiplicity of multiple lines we construct in this case the perturbed systems corresponding to the given canonical forms associated to the configurations of invariant lines.

In **Chapter 4** we state and prove the classification theorem (Main Theorem D) of the cubic systems in $\mathbb{CSL}_8$ with two distinct $\mathbb{ISP}s$. For this family of systems we construct all possible configurations of invariant lines and affine invariant criteria for the realization of each one of the detected configurations. Surely we construct in this case the perturbed systems corresponding to the given canonical forms associated to the configurations of invariant lines. Besides the proof of Main Theorem D we detect a new class of systems in $\mathbb{CSL}_9$ which completes the classification given by Llibre and Vulpe in [83].

1. ADVANCES IN THE STUDY OF CUBIC SYSTEMS WITH INVARIANT STRAIGHT LINES

We consider real planar polynomial differential systems, i.e. systems of the form

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y) \tag{1.1}$$

where p and q are polynomials in x and y with real coefficients and $\max\big(\deg P, \deg Q\big) = n$. We call such systems *polynomial systems of degree n.* In particular, a planar polynomial system of differential equations of degree 2 or 3 will be called simply, a quadratic system or a cubic system, respectively. We denote by $\mathbb{CS}$ the whole class of real cubic differential systems. To a system (1.1) one can associate the vector field $\mathbb{X} = P(x, y)\dfrac{\partial}{\partial x} + Q(x, y)\dfrac{\partial}{\partial y}$.

1.1. Polynomial differential systems with invariant straight lines

The polynomial differential systems are objects of numerous scientific investigations. For reasons, which have, apparently, philosophical character, the majority of differential equations born in engineering practice, are not integrable. More precisely, they cannot be solved in quadrature, i.e. it is not possible to reduce the problem to integration of known functions. A source of these difficulties is nonlinearity of mathematical models of complicated real processes and devices. Poincaré wrote a series of memoirs [89–91] where he built a new branch of mathematics, called *qualitative theory of differential equations.* More precisely, even if the differential equation can not be solved in terms of known functions, yet from the very form of the equation, a wealth of information about the geometric properties and behavior of the solutions can be found. In other words, the solutions of the differential equation are a family of functions. Graphically, this can be plotted in the phase plane like a two-dimensional vector field. Vectors representing the derivatives of the points with respect to a parameter (say time t), that is $(dx/dt, dy/dt)$, at representative points are drawn. With enough of these arrows in place the system behavior over the regions of plane in analysis can be visualized and limit cycles can be easily identified. A phase portrait is a geometric representation of the trajectories of a differential system in the phase plane. Phase portraits are an invaluable tool in studying dynamical systems. They consist of a plot of typical trajectories (invariant algebraic curves) in the state space.

These curves or trajectories can be defined on the Poincareé disc using the Poincaré compactification [113], which is define as follows. Consider the x, y plane as being the plane $Z = 1$ in the space $\mathbb{R}^3$ with coordinates X, Y, Z. The central projection of the vector field

$\mathbb{X}$ on the sphere of radius one yields a diffeomorphic vector field on the upper hemisphere and also another vector field on the lower hemisphere. Poincaré indicated briefly in [90] that one can construct an analytic vector field $\mathcal{V}$ on the whole sphere such that its restriction on the upper hemisphere has the same phase curves as the one induced by the phase curves of (1.1) via the central projection. A complete proof was given much later in [66]. The analytic vector field $\mathcal{V}$ on the whole sphere obtained in this way is called the Poincaré field associated to a system (1.1). The phase curves of $\mathcal{V}$ coincide in each chart with phase curves induced by planar polynomial vector fields, in particular in the chart corresponding to $Z = 1$, denoting the two coordinate axes x, y corresponding to the OX and OY directions, they coincide with the phase curves induced by (1.1). The two planar polynomial vector fields U, V associated to the charts for $X = 1$ (with local coordinates (u, z)) and for $Y = 1$ $\big($with local coordinates $(v, w)\big)$ and changes of coordinates $u = y/x, z = 1/x$, or $v = x/y, w = 1/y$ are as follows.

$$U: \ \frac{du}{dt} = C^\star(1, u, z), \ \frac{dz}{dt} = zP^\star(1, u, z); \ V: \ \frac{dv}{dt} = C^\star(v, 1, w), \ \frac{dw}{dt} = -wQ^\star(v, 1, w),$$

where $P^\star(X, Y, Z), Q^\star(X, Y, Z)$ are the homogeneous polynomials associated to the polynomials $P(x, y), Q(x, y)$. By the compactification of the planar polynomial vector field associated to (S) we understand the restriction $V|_{\mathcal{H}^1}$ (where by $\mathcal{H}^1$ we understand the upper hemisphere $\mathcal{H}$ completed with the equator) of the analytic vector field $\mathcal{V}$ on the sphere.
Our aim of the Thesis is to classify $\mathbb{CS}$ according to their geometric proprieties, i.e. in the construction of the configurations of invariant straight lines of these systems on $\mathbb{R}^2$ completed with its points "at infinity", i.e. on the equator S^1 of S^2. Since the vertical projection is a diffeomorphism of $\mathcal{H}^1$ on the disk $\{(x, y)|x^2 + y^2 \leq 1\}$ we can view the such configurations of our systems (1.1) on this disk, called *the Poincaré disk*.

In this Thesis are approached the planar differential cubic systems. The set $\mathbb{CS}$ depends on 20 parameters and for this reason people began by studying particular subclasses of $\mathbb{CS}$. We are interested in the investigation of the cubic polynomial differential systems with invariant straight lines.

Following [114, Schlomiuk], we call *configuration of invariant straight lines* of these systems, the set of (complex) invariant straight lines (which may have real coefficients) of the system, each endowed with its own multiplicity and together with all the real singular points of this system located on these invariant straight lines, each one endowed with its own multiplicity.

It was observed (see, for instance [85,86,103,104,110,112,127,128]) that the configurations of invariant straight lines which were detected for various families of systems (1.1) using Poincaré compactification, could serve as a base to complete the whole Poincaré disc with the trajectories of the solutions of corresponding systems, i.e. to give a full topological classification of such systems.

According to [5] the maximum number of invariant straight lines taking into account their multiplicities for a polynomial differential system of degree m is $3m$ when we also consider the infinite straight line. This bound is always reached if we consider the real and the complex invariant straight lines, see [43]. So the maximum number of the invariant straight lines (including the line at infinity $Z = 0$) for cubic systems is 9.

In [5] some basic results on the invariant straight lines of a polynomial differential system of degree n were given. We emphasis these properties for $n = 3$, i.e. for the case of a real differential cubic system in $\mathbb{CS}$:

(a) either all the points on an invariant straight line are singular or the line contains no more than 3 singular points;

(b) no more than 3 invariant straight lines can be parallel;

(c) the set of all invariant straight lines passing through a single point cannot have more than 4 different slopes;

(d) either it has infinitely many finite singular points, or it has at most 9 finite singular points.

At infinity a system in $\mathbb{CS}$ has at most four distinct singular points (in the Poincaré compactification) if $C_3(x, y) = yP_3(x, y) - xQ_3(x, y) \neq 0$ and for a system in $\mathbb{CS}$ the infinity represents a non-singular invariant straight line. In the case $C_3(x, y) = 0$ the infinity is degenerate, i.e. consists only of singular points. According to [4] (see also [130]) systems in $\mathbb{CS}$ with the degenerate line at infinity possess at most six invariant lines. Since in the Thesis we investigate only systems in $\mathbb{CS}$ with invariant lines of total multiplicity eight, we shall focus only on the case $C_3 \neq 0$ when the systems have the non-degenerate infinity.

Over the years many researchers have approached various problems concerning the qualitative study of polynomial systems. Since we are interested on cubic systems possessing invariant straight lines, we shall give here below a survey of some of results in this area in the case of arbitrary n and also for the simplest nonlinear systems (1.1) with $n = 2$ and $n = 3$. We underline that in the case of cubic systems we will give a more detailed review because this is the domain of our Thesis and, of course, we will compare the results and point out what are the main important distinctions with respect to the results obtained till now by other authors.

In the papers by Popa and Sibirski [96,97,100, 1987-1992] the conditions for the existence and the number of invariant straight lines of a system (1.1) with homogeneous nonlinearities of degree $n > 1$, as well as the conditions with the maximal multiplicity of an invariant line were determined.

Robert Kooij [71, 1995] considered polynomial systems (1.1) with $n + 1$ invariant lines. As a particular result it was proved that under some circumstances there are no limit cycles, whereas in other situations it is shown that limit cycles do exist.

In the paper [123, 1996] Sokulski has studied the problem of the existence of invariant straight lines of planar polynomial systems (1.1). In particular for $n = 4$ he has shown that the maximum number of isolated real invariant straigt lines of such systems is 9.

In [139, 1993] Zhang Xi Kang has proposed a conjecture regarding the maximum number of invariant straight lines for systems (1.1) of degree n (this conjecture was proved for $n = 3$ and $n = 4$). However the author has committed a mistake which he has corrected in [140, 1998].

The class of all real polynomial systems (1.1) with $n > 1$ and with a finite number of invariant straight lines is considered by Artes, Llibre and Grünbaum in [4, 1996] and [5, 1998]. The authors estimated the maximum number of such lines, possible in this class, as well as the number of slops of invariant lines for systems (1.1).

The Chinese mathematician Dai Guo Ren [55, 1996] also has considered polynomial systems (1.1) with invariant lines. He has estimated the number of non-parallel invariant lines for $n \geq 3$ as well as the number of parallel ones in the case $n \geq 2$.

In one of the most recent publication [61, 2015] Dolov and Kruglov have obtained a sharp bound for the number of distinct invariant straight lines of non-degenerate systems (1.1) with $n \geq 2$. However we have to point out that this result have been obtained earlier by other authors and namely, in the paper [5]. Moreover this result was generalized by Llibre and Medrado in the article [81, 2007], where the number of invariant hyperplanes for d-dimensional polynomial vector fields was given.

A lot of papers are dedicated to the polynomials (1.1) possessing invariant lines in the simplest case of these systems, i.e. in the case $n = 2$. We give here only several publications which in our opinion are relevant.

In 1991 Popa and Sibirski [100, 101], using the invariant polynomials of quadratic systems (1.1) with respect to linear group of transformations $GL(2, \mathbb{R})$, have constructed the necessary and sufficient conditions for the existence of a non-homogeneous invariant affine line as well as for the number of homogeneous invariant lines.

Voldman and Vulpe [133, 1999] have constructed affine invariant coefficient conditions for a quadratic system to possess two conjugate imaginary invariant straight line as well as two couples of such invariant lines. Moreover, for some classes of quadratic systems with two imaginary invariant straight lines necessary and sufficient affine invariant conditions for possessing just one limit cycle are determined.

Between 2004 and 2015 there were published a set of articles by Schlomiuk and Vulpe dedicated to the family of planar quadratic systems (1.1) possessing invariant lines of total multiplicity greater then or equal to three. First we mention the articles [108, 111, 114] in which for this family are constructed all the possible configurations of invariant lines of given total multiplicity. Moreover applying the invariant theory of differential systems the necessary and sufficient conditions for the realization of each one of constructed configurations are determined.

Other set of papers [110, 112, 113, 115, 116] deals with the above mentioned families of systems, but solving other problems, and namely: i) the construction of the corresponding first integrals and ii) the global topological classification of these systems. We remark that in this case the necessary and sufficient affine invariant conditions for the realization of each one of the detected phase portraits are established.

In what follows we shall focus on systems in $\mathbb{CS}$ and we shall describe the current situation in this domain.

The Russian mathematician Liubimova in [85, 1977] showed that cubic systems (1.1) could possess at most 8 real distinct invariant affine lines. Moreover she has investigated cubic systems with the maximum number of such lines and constructed the corresponding canonical form (which turned out to be a specific system with numerical coefficients), as well as its phase portrait. The paper [86, 1984] is a continuation of [85] and it is devoted to cubic systems with 7 distinct real invariant affine straight lines. The author also has constructed three canonical forms depending on the number of triplets of parallel invariant lines. As a result there were determined four topologically distinct phase portraits.

The mathematicians from Republic of Moldova Cozma and Şubă also have considered the family of cubic systems possessing invariant straight lines (see [50–54, 124–126, 1992-2005]). More exactly, in these papers they have investigated a special family of cubic systems (1.1) having at the origin of coordinates a singularity $O(0,0)$ with pure imaginary eigenvalues.

In addition they force the systems to possess three or four invariant straight lines and constructed the necessary and sufficient conditions for the singularity $O(0,0)$ to be a center. Cozma have generalized this problem in [45–49, 2009-2012] considering cubic systems with the singularity $O(0,0)$ of the same type, which besides two invariant straight lines possess one invariant conic. As a result for such systems the center problem was solved and this leads to the construction of some new conditions for the existence of a center for cubic systems.

A group of mathematicians, Llibre, Mahdi, Vulpe (form Spain, Turkey and Republic of Moldova), have considered cubic systems (1.1) having a quadratic rational first integral [80, 2011]. It turned out that such systems possess invariant straight lines which play a central role in the topological classification of these systems. The authors have proved the existence of 38 topologically distinct phase portraits (among which 11 correspond to degenerate systems).

The Chinese mathematician Chan Guo Wei has considered some families of cubic systems possessing either two invariant conjugate imaginary lines [38, 1991] or two pairs of such conjugate lines [39, 1995] [40, 1997] or just two real intersecting invariant straight lines [41, 1998]. In these papers the author has investigated the problem of the existence or the non-existence of limit cycles. In particular, in the last paper, it is shown that a cubic system (1.1) has no limit cycles when the two real invariant straight lines intersect each other. On the other hand necessary and sufficient conditions for the existence of a unique limit cycle are given in the case of two parallel invariant straight lines.

Cubic systems with either six or seven invariant straight lines are also investigated by mathematicians Şubă, Puţuntică and Repeşco [103, 104, 127, 128]. In these papers they used the notion of *parallel multiplicity*. The authors say that an invariant line $f(x,y)=0$ where $f(x,y) = ux+vy+w$ of a cubic system (1) has parallel multiplicity $1 \le k \le 3$ if the identity $\mathbb{X}(f) = f^k R(x,y)$ holds for some polynomial $R(x,y)$ with coefficients in $\mathbb{C}$ (here $\mathbb{X}$ are the vector fields associated to systems (1.1)). If in the case of cubic systems with seven invariant straight lines one also takes into account the line at infinity as invariant line, then in fact the authors have considered the family of cubic systems possessing configurations of invariant straight lines of total parallel multiplicity 8. Considering the whole family of cubic systems authors proved that there exist exactly 17 configurations of invariant lines. These configurations coincide with the configurations detected in this Thesis but only for the subfamily of cubic systems possessing four distinct infinite singularities (see Chapter 2). This coincidence is a natural one because it is clear that a system in such a family could not have invariant straight line of, say, "non-parallel" multiplicity, as for these lines we must have at least one infinite singular point defined by a multiple linear factor of the form $C_3(x,y) = yp_3(x,y) - xq_3(x,y)$ (when we factorize C_3 over $\mathbb{C}$). The case of singularities at infinity defined by multiple factors of C_3 are considered in Chapters 3 and 4 and the constructed configurations could not be obtained applying only the notion of "parallel" multiplicity.

We would like also to mention one more difference between the results obtained in the classification given by these authors and those from our Thesis (for the above mentioned subfamily). And namely, in contrast to the results obtained in the papers [127] and [128], in our Thesis in addition we find out the necessary and sufficient conditions for the realization of each one of the 17 configurations.

Finally we would like to discuss some results obtained by Llibre and Vulpe [83, 2006] which strongly correlate with the results contained in our Thesis. More precisely in [83] the cubic systems (1.1) possessing the maximum number of invariant straight lines were considered. The authors introduced the notion of the configurations of invariant lines of cubic systems and detected 23 such configurations for cubic systems. Moreover using invariant polynomials with respect to the group $Aff(2,\mathbb{R})$ of affine transformations and time rescaling in this paper the necessary and sufficient conditions for the realization of each one of 23 configurations were constructed. However, as it was proved in this Thesis (see also [16, 2014]), the classification given in [83] is not complete. Indeed, in Chapter 4 of the Thesis we exhibit a new class of cubic systems with invariant lines of total multiplicity 9.

We would like to underline that for some special families of polynomial systems (1.1) possessing invariant lines, the knowledge of configurations of lines allows us easily to detect the corresponding phase portraits. For example, in papers [110, 112] for quadratic systems with invariant lines grater or equal to 4, it was proved that the existence of 57 distinct configurations of invariant lines leads to the existence of 135 topologically distinct phase portraits. In [103, 104, 127, 128] for cubic systems with invariant lines of total parallel multiplicity six or seven, taking into consideration constructed configurations of invariant lines it was proved the existence of 113 topologically distinct phase portraits.

In the Thesis for systems $\mathbb{CSL}_8$ there have been obtained 51 distinct configurations of invariant straight lines, this number summarizing all possible configurations obtained in all cases examining in Chapters 2-4.

The question of determining all topologically distinct phase portraits of systems in $\mathbb{CSL}_8$ is expected to be examined by the author in the future.

1.2. Invariant algebraic curves in the study of integrability of planar polynomial systems

As it was mentioned earlier, one of the goal of the curent Thesis is the determination of all configurations of invariant straight lines for systems in $\mathbb{CSL}_8$. Therefore, in order to construct these configurations, we have to determine all possible invariant straight lines, the sum of multiplicities of which equals eight. Our hypothesis is that these invariant lines could serve as a base for determining the first integral (integrating factor) of corresponding systems applying the method of integration of Darboux. In what follows we shall examine this point of view.

The question to determine the invariant algebraic curves of a given planar polynomial vector field, or to decide that no such curves exist, is part of a problem set forth by Poincaré, and is also essential in deciding whether the vector field is integrable (admits an integrating factor). See the interesting and profound survey of Schlomiuk [105] on these questions. One of the first publications in this direction was the paper of Darboux [59, 1878] where the existence of invariant curves plays an important role in order to determine the first integrals of these systems. Since Darboux had found connections between algebraic geometry and the existence of first integrals of polynomial systems, invariant algebraic curves have been a central object in the theory of integrability of planar polynomial systems. Today, after more than a century of investigations, the theory of invariant algebraic curves is still full of open questions which are not easy to solve. One of the reasons for this is the fact that it is very difficult to tell if any nonsingular trajectories of the system are contained in algebraic curves.

The method of integration of Darboux uses multiple-valued complex functions of the form: (a) $F = e^{G(x,y)} f_1(x,y)^{\lambda_1} \cdots f_s(x,y)^{\lambda_s}$, $G = G_1/G_2$, $G_i \in \mathbb{C}[x,y]$, and f_i irreducible over $\mathbb{C}$. It is clear that in general the last expression makes sense only for $G_2 \neq 0$ and for points $(x,y) \in \mathbb{C}^2 \setminus \big(\{G_2(x,y)=0\} \cup \{f_1(x,y)=0\} \cup \cdots \cup \{f_s(x,y)=0\}\big)$.

Consider the polynomial system of differential equations (1.1). The equation $f(x,y)=0$ ($f \in \mathbb{C}[x,y]$, where $\mathbb{C}[x,y]$ denotes the ring of polynomials in two variables x and y with complex coefficients) which describe implicitly trajectories of of systems (1.1), can be seen as an affine representation of an algebraic curve of degree m. Suppose that (1.1) has a solution curve which is not a singular point, contained in an algebraic curve $f(x,y)=0$. It is clear that the derivative of f with respect to t must vanish on the algebraic curve $f(x,y)=0$, so

$\frac{df}{dt}|_{f=0} = (\frac{df}{\partial x}P(x,y) + \frac{df}{\partial y}Q(x,y))|_{f=0} = 0.$

In 1878 Darboux introduced the notion of the invariant algebraic curve for differential equations on the complex projective plane. This notion can be adapted for systems (1.1). According to [59] an algebraic curve $f(x,y) = 0$ in $\mathbb{C}^2$ with $f \in \mathbb{C}[x,y]$ is an *invariant algebraic curve* (an algebraic partial integral) of a polynomial system (1.1) if $\mathbb{X}(f) = fK$ for some polynomial $K(x,y) \in \mathbb{C}[x,y]$ called the cofactor of the invariant algebraic curve $f(x,y) = 0$. It could be observed that for the points of the curve $f(x,y) = 0$ the right hand side of (1.1) is zero. This means that the gradient $(\partial f/\partial x, \partial f/\partial y)$ is orthogonal to the vector field $\mathbb{X} = (P,Q)$ at these points. Therefore the vector field $\mathbb{X}$ is tangent to the curve $f = 0$. This explains why the algebraic curve $f = 0$ is invariant under the flow of the vector field $\mathbb{X}$.

In view of Darboux's definition, an algebraic solution of an equation (1.1) is an invariant algebraic curve $f(x,y) = 0$, $f \in \mathbb{C}[x,y]$ $(\deg f \geq 1)$ with f an irreducible polynomial over $\mathbb{C}$. Darboux showed that if a system (1.1) possesses a sufficient number of such invariant algebraic solutions $f_i(x,y) = 0$, $f_i \in \mathbb{C}$, $i = 1,2,\ldots,s$, then the system has a first integral of the form (a) (see the previous page).

According to [59], we say that a system (1.1) has a Darboux first integral (respectively Darboux integrating factor) if it admits a first integral (respectively integrating factor) of the form $e^{G(x,y)}\prod_{i=1}^{s} f_i(x,y)^{\lambda_i}$, where $G(x,y) \in \mathbb{C}(x,y)$ and $f_i \in \mathbb{C}[x,y]$, $\deg f_i \geq 1$, $i = 1,2,\ldots,s$, f_i irreducible over $\mathbb{C}$ and $\lambda_i \in \mathbb{C}$. If a system (1.1) has an integrating factor (or first integral) of the form $F = \prod_{i=1}^{s} f_i^{\lambda_i}$ then $\forall i \in \{1,\ldots,s\}$, $f_i = 0$ is an algebraic invariant curve of (1.1).

In [59] Darboux proved the following remarkable theorem of integrability using invariant algebraic solutions of systems (1.1):

Theorem. Consider a differential equation (1.1) with $p,q \in \mathbb{C}[x,y]$. Let us assume that $m = \max(\deg P, \deg Q)$ and that the equation admits s algebraic solutions $f_i(x,y) = 0$, $i = 1,2,\ldots,s$ $(\deg f_i \geq 1)$. Then we have:

I. If $s = m(m+1)/2$ then there exists $\lambda = (\lambda_1,\ldots,\lambda_s) \in \mathbb{C}^s \setminus \{0\}$ such that $R = \prod_{i=1}^{s} f_i(x,y)^{\lambda_i}$ is an integrating factor of (1.1).

II. If $s \geq m(m+1)/2 + 1$ then there exists $\lambda = (\lambda_1,\ldots,\lambda_s) \in \mathbb{C}^s \setminus \{0\}$ such that $F = \prod_{i=1}^{s} f_i(x,y)^{\lambda_i}$ is a first integral of (1.1).

In 1979 Jouanolou proved the next theorem which improves part II of Darboux's Theorem: Consider a polynomial differential equation (1.1) over $\mathbb{C}$ and assume that it has s algebraic solutions $f_i(x,y) = 0$, $i = 1,2,\dots,s$ ($\deg f_i \geq 1$). Suppose that $s \geq m(m+1)/2+2$. Then there exists $(n_1,\dots,n_s) \in \mathbb{Z}^s \setminus \{0\}$ such that $F = \prod_{i=1}^{s} f_i(x,y)^{n_i}$ is a first integral of (1.1). In this case $F \in \mathbb{C}(x,y)$, i.e. F is rational function over $\mathbb{C}$.

The above theorems shows that we can reduce the study of the invariant algebraic curves, to the study of the irreducible invariant algebraic curves in $\mathbb{C}[x,y]$, (see, for instance Christopher and Llibre [42]).

According to the above stated theorems, if a system in $\mathbb{CS}$ possesses 7 distinct affine invariant straight lines (algebraic curves of the first degree) then this systems has a first integral which could be constructed with these lines and so, the coordinates of the vector λ could easily be determined. Moreover, if cubic systems (1.1) possess 6 distinct affine invariant lines, then it exists an integrating factor determined by these lines.

As it was mentioned above the main object of our investigations is the class $\mathbb{CSL}_8$ of cubic systems, which is, at the first glance, very specific one. The next question which comes to mind is the following: could a system (1.1) in $\mathbb{CSL}_8$ having ≤ 5 distinct affine invariant straight lines of total multiplicity 8 be Darboux integrable? Our assumption concerning this problem is that a first integral (or an integrating factor) also could be constructed for considerated systems.

This conviction is based on the fact that in [127, 128, Şubă, Repeşco, Puţuntică](also see [85, 85]) the authors have been proved that cubic systems (1.1) with invariant lines of parallel multiplicity 7 are integrable. Particularly a first integral for cubic systems with four distinct affine invariant lines is constructed.

It is worth to point out that the problem of integrability in the case of cubic systems with the maximum number of invariant straight lines (i.e. 9), considered in [83], is still unsolved.

Thus, we arrive at the conclusion that, having all canonical forms of systems in $\mathbb{CSL}_8$, constructed in the current Thesis, the problem of integrability of such systems could be solved. This question is expected to be examined by the author in the future.

1.3. Invariant theory of polynomial differential systems in the problem of classification

The roots of the invariant theory of polynomial vector fields lie in the classical invariant theory. The idea to adapt to polynomial vector fields the concepts of classical invariant theory is due to C.S. Sibirsky, the founder of the Chişinău school. In this chapter we first describe briefly classical invariant theory and then explain how the ideas of this theory were used by Sibirsky and his school to built an analogous theory for polynomial vector fields.

Classical invariant theory. In the Introduction of his book [87], Olver defines classical invariant theory as *the study of the intrinsic or geometrical properties of polynomials.* By *geometric property* we mean a property which is not affected by changes of coordinates. These changes are usually assembled in groups such as for example the group $GL(n,\mathbb{C})$ of linear transformations of $\mathbb{C}^n$ or its subgroup $SL(n,\mathbb{C})$ of special linear transformations of determinant 1. In his famous Erlangen Program, Felix Klein called a property *geometric* if it is invariant under a group action. Depending on the groups we consider we could have the Euclidean geometry for the group of rigid motions, the affine geometry for the group of affine transformations, the projective geometry for the group $PGL(3,\mathbb{C})$ of projective transformations.

We can trace the beginning of classical invariant theory to Gauss who in 1801 published in Latin his book Disquisitiones arithmeticae where Gauss observed invariant behavior in the theory of quadratic forms over $\mathbb{C}$. Gauss considered binary quadratic forms $Q(x,y) = ax^2 + 2bxy + cy^2$ over $\mathbb{C}$. On these forms acts the group $GL(2,\mathbb{C})$ of linear transformations, i.e. for every 2×2 non-singular matrix M over $\mathbb{C}$ and every quadratic form $Q(x,y)$ we can associate a quadratic form $\tilde{Q}(\tilde{x},\tilde{y}) = A\tilde{x}^2 + 2B\tilde{x}\tilde{y} + C\tilde{y}^2$ where $(x,y)^t = M(\tilde{x},\tilde{y})^t$. The discriminant of a quadratic form Q over $\mathbb{C}$ is the number $\Delta = b^2 - ac$. For the transformed form we have $\tilde{\Delta} = B^2 - AC$ and calculations yield $\tilde{\Delta} = (detM)^2\Delta = C\Delta$ with $C = (detM)^2 \in \mathbb{C}$, C depending only on the element M of the group $GL(2,\mathbb{R})$ and not on the quadratic form. If $det(M) = 1$ then Δ is an invariant. In [64] Gauss was interested in number theory and used quadratic forms over $\mathbb{Z}$ with the action of the group $SL(2,\mathbb{Z})$ on these forms.

The classical theory of invariants deals with transformations of forms, i.e. homogeneous polynomials in n variables $x = (x_1, ..., x_n)$ of degree m with coefficients in $\mathbb{C}$. In particular, for the specials case of binary forms we have $f(a,x) = \sum_{i=0}^m \begin{pmatrix} m \\ i \end{pmatrix} a_i x_1^{m-i} x_2^i$ with $a = (a_1, ..., a_n)$. If $x = Mx'$ is a non-singular linear transformation, then this induces a transformation of f, i.e. $f(a,x) = f(a,Mx') = f'(a',x')$ where f' is a new n-form with coefficients $a' = (a'_1, ..., a'_n)$. Then the coefficients a' of f' are related to those of f by a linear transformation $a = \nu(M)a'$. Furthermore if $M = M_1M_2$ then $\nu(M) = \nu(M_1M_2) = \nu(M_1)\nu(M_2)$, so the map $M \to \nu M$ is an automomorphism of the group $GL(n,\mathbb{C})$. If we only consider M in a subgroup G of $GL(n,\mathbb{C})$ then this map is a group homomorphism $G \to GL(n,\mathbb{C})$. As Thomas Hawking observed in [69], this property was ignored in the first phase of the development of classical invariant theory but it was later emphasized by Lie and by Klein in his Erlangen program of 1872. In fact there is a close relation between invariant theory and group theory. In his book [60] Dieudonné talking about the revival prompted by developments of Schur, Weyl and Cartan, on the semi-simple groups and their representations around 1935, *it was realized that classical invariant theory was really a special case of that theory."* In modern terms the invariants of the form f are called invariants of the representation $M \to \nu(M)$ in $GL(m+1,\mathbb{C})$ induced by the action of $GL(n,\mathbb{C})$ on $\mathbb{C}^n$.

A polynomial $I(a)$ in the a_i is a *relative invariant of f (or of a family of such forms f)* if $I(a') = (det(M))^{-\mu} I(a)$ for all $M \in GL(2,\mathbb{C})$ and μ is called the weight. In case $I(a') = I(a)$ then I is called an *invariant* (or an *absolute invariant*).

A polynomial $g(a,x)$ in the a_i and x_j of $f(a,x)$ is called a *relative covariant* of f if $g'(a',x') = (detM)^{\nu} g(a,x)$ for all $M \in GL(2,\mathbb{C})$. The classical theory of invariants is the study of these invariants and covariants of n-form over $\mathbb{C}$ and finding all of them.

The classical invariant theory began to be developed in the 1840s. The British mathematician George Boole launched this theory by publishing the treatise "Exposition of a General Theory of Linear Transformations" in 1841. The basic example considered by Boole was the set of monic polynomials $Q(x) = x^2 + 2b_1x + b_2$ on which acts the group of translations, $Q(x+c) = x^2 + 2(b_1+c)x + c^2 + 2b_1c + b_2$. The discriminant of $Q(x)$ is $\Delta = b_1^2 - b_2$ and calculations yield that the discriminant Δ' of $Q(x+c)$ coincides with Δ. Thus the discriminant is an invariant of monic quadratic polynomials under the action of translations. Boole used elimination theory to construct an invariant of homogeneous polynomials of degree m in n variables, namely the discriminant K. In 1845, in [35] Cayley gave a technique for generating invariants, distinct from that of Boole. Cayley's goal was to elaborate a theory that would allow for the production of some sort of minimal set of invariants for a given form and also of a minimal set of relations or syzygies among invariants. For example Cayley constructed two simpler invariants I, J than the ones of Boole and it turned out that K, I, J are related by the "syzygy" $K = I^3 - 27J^2$. The term "syzygy" was introduced in 1853 by James Joseph Sylvester who was also a leader of the British school of invariants.

The German school led by Otto Hesse, included his student S. Aronhold, also A. Clebsch and P. Gordan. Paul Gordan proved the first fundamental theorem of invariant theory in 1868 in [67]. Gordan considered only binary forms over a field K under the action of the group $GL(2,K)$. His theorem says that *all covariants are explicitely constructible as polynomials in a finite number of them over the underlying field.* The syzygy problem, i.e.

find a finite basis for syzygies, remained unsolved.

For over twenty years, extending the theorem of Gordan to n-ary forms proved to be a difficult task but in 1890 Hilbert, who obtained his doctorate in 1885 with a thesis on invariant theory and with an Habilitationsschrift (1886) also on invariant theory, succeeded in generalizing the result of Gordan to n-ary forms over $\mathbb{C}$, i.e. homogeneous polynomials in n variables $F(x_1, ..., x_n)$ with coefficients in $\mathbb{C}$, of any degree. Hilbert's first proof of this result was existential, i.e. he proved the existence of a finite basis of invariants by reductio ad absurdum. It was only in 1893 that he gave a constructive proof for this theorem which says that any finite system of homogeneous polynomials admits a basis for its invariants, as well as for its covariants. A collection of invariants $I_1, ..., I_s$ of n-forms $f_1, ..., f_l$ of degree m such that any invariant I is a polynomial function $I = P(I_1, ..., I_s)$ is called a Hilbert basis. Similarly a finite collection of covariants $J_1, ..., J_k$ forms a Hilbert basis if every covariant J is a polynomial function $J = P(J_1, ..., J_k)$.

In 1893 Hilbert solved also the problem of finding a finite basis for syzygies of invariants of n-ary forms. Hilbert's work went beyond invariant theory since in these papers he proved results of modern algebra which are now at the basis of commutative algebra: Hilbert's basis theorem and the Nullstellensatz. Thus invariant theory played an important role in the development of abstract algebra at the beginning of the twentieth century.

In 1900, at the International Congress of Mathematicians in Paris, Hilbert included in his list of problems for the XX-th century [76] also a problem on invariant theory. This is his 14-th problem on his list. However, after his work [75], Hilbert abandoned invariant theory and with few exceptions, this theory remained rather stagnant until the middle of the twentieth century when it was revived and infused with new ideas ultimately giving birth to modern invariant theory or geometric invariant theory as it was called by Mumford [70].

We note that we have a relation between classical invariant theory and group representation theory, relation which was not perceived in the initial stages of invariant theory but which was highlighted in the work of Lie and Klein. This relation is pervasive in modern invariant theory.

Sibirsky's school in invariant theory of planar polynomial vector fields. Classical invariant theory involved vector spaces of n-forms over $\mathbb{C}$ of degree m, where n and m are natural numbers. It was a natural problem to try to develop an analogous theory in which the vector space of n-forms is replaced by a vector space of differential equations such as for example the vector space of polynomial differential systems of degree m. Although algebraic invariants in the theory of differential equations were used in work of some 19th century mathematicians, the development of a full invariant theory for polynomial differential systems only began in the second part of the XXth century. The initial steps in this direction were made by Constantin Sibirsky who wrote several books and published numerous works on this theme. He established the connection between the algebraic invariants of the classical groups (rotation groups, $O(2,\mathbb{R})$, $GL(2,\mathbb{R})$, $Aff(2,\mathbb{R})$) and the qualitative theory of planar polynomial differential systems. Sibirsky was the founder of the Chişinău school in the theory of invariants of planar polynomial vector fields. He had many students, among them V. Baltag, D. Bularas, I. Calin, Dang Din' Bik, V.I. Daniljuc, F. Gasinskaja-Kirnickaja, V. Lunchevich, A.V. Marincuk, I.I. Pleshkan, M. Popa, A.M. Stakhi, A. Şubă, V.D. Taku, N. Vulpe. Most of them had students and they continue to be active.

In his book [121], Sibirskii wrote that the first works on the theory of invariants of differential equations were done by Laguerre. His papers, published in 1879, are on linear differential equations. From 1879 on, other articles on invariants of linear differential equations were published, among them the work of Halphen. Beginning with the work of Liouville (1886) and of Appell (1889), the theory of invariants was extended to non-linear differential equations and new contacts between this theory and algebraic geometry appeared (see [121] for more information).

Sibirsky began to work on developing the invariant theory of polynomial vector fields during the 1950's. The first successful articles on classifying families of planar polynomial systems were done by Sibirsky and Vulpe, one published in 1975 [137] and another in 1977 [138]. A few unsuccessful attempts to classify quadratic systems with a center were done before 1975. Vulpe's article [135] published in 1983 contained the first correct classification for this important class of all quadratic systems with a center. Vulpe not only listed there all 31 phase portraits of this family but also gave necessary and sufficient conditions in terms of invariant polynomial for the realization of each one of the phase portraits, when the center is placed at the origin. Thirty two years have passed since the publication of this article and in this period of time over 50 articles appeared solving classification problems for quadratic or cubic differential systems. These articles are of two kinds: (A) those listing phase portraits but without proving with the help of topological invariants that the lists contain topologically distinct phase portraits and without characterizing the phase portraits (whenever possible) in terms of invariant polynomials (for this concept see the next section); (B) those proving that the phase portraits are indeed distinct and characterizing each phase portrait in terms of polynomial invariants (see [117]) whenever this was possible. Here below we give a very brief survey of results obtained by Sibirsky and by members of the Chişinău school and their collaborators.

With his students, Sibirsky constructed algebraic invariants and comitants (analogs in this theory, of the covariants in the classical theory) for polynomial vector fields under the action of the groups $O(2, \mathbb{R})$, $GL(2, \mathbb{R})$ and $Aff(2, \mathbb{R})$.

The basic problem of giving polynomial bases for invariants and comitants for quadratic and cubic differentials systems for the action of the group $GL(2, \mathbb{R})$ was solved by Sibirsky and his students Bularas, Calin, Daniljuc, Dan Ging'Bik, Gasinskaja-Kirnickaja and Vulpe (see [33, 56, 57, 63, 134]). In addition to the construction of polynomial bases, Calin constructed algorithms for creating rational bases of the $GL(2, \mathbb{R})$-comitants for polynomial differential systems (see [37, 44]).

The problem of giving polynomial bases for invariants and comitants for the quadratic differential systems with respect to the group $Aff(2, \mathbb{R})$ was solved by Bularas in [31], [32]. He obtained 36 affine comitants such that every invariant or comitant with respect to $Aff(2, \mathbb{R})$ is obtained as a polynomial in these 36 elements. Using these elements as well as the elements of polynomial bases of $GL(2, \mathbb{R})$-comitants, in [34], the minimal polynomial basis of T-comitants (see definition in Section 2.1) was constructed up to and including the degree 12.

For cubic differentials systems the problem of giving polynomial bases for invariants and comitants, for the action of the group $Aff(2, \mathbb{R})$ is still unsolved.

These bases play a fundamental role in the study of families of quadratic or cubic differential systems. In particular they were used for the purpose of classifying families of polynomial differential systems. This work was an important contribution because, unlike other classification results on these systems which were non-intrinsic, the results obtained by Sibirskii and his students were intrinsic, i.e. they were invariant under allowable coordinate changes and hence independent of the specific presentation of the systems (normal forms) used in the classifications.

Sibirsky determined necessary and sufficient conditions in terms of algebraic invariants with respect to the $GL(2, \mathbb{R})$ action, for a system with only quadratic or cubic nonlinearities to have a center at the origin (see [118], [119]). Later Sibirsky with Kirnickaja have constructed analogous conditions for the existence of two centers, one of which being placed at the origin.

For a general quadratic system in the plane (including a system in which there can be free terms) Bularas, Vulpe and Sibirsky obtained necessary and sufficient conditions for the existence of one center, and also of two centers. These conditions are expressed in the form of equalities and inequalities between affine-invariant polynomials in the coefficients of the right-hand sides of the system.

The Poincaré-Lyapunov constants intervene in the problem stated by Poincaré of distinguishing a focus from a center (see [89]). Calin worked on the expression of the general Poincaré-Lyapunov constants in terms of invariants of the $GL(2,\mathbb{R})$ group action. (In this case the singular point is placed at the origin.)

Another theme on which Sibirsky and his former student Popa worked was the problem of determining $GL(2,\mathbb{R})$ invariant necessary and sufficient conditions for the existence of an invariant straight line [100]. This theme was further pursued by Schlomiuk and Vulpe in a series of papers where they gave: i) necessary and sufficient conditions, in terms of affine invariant polynomials, for a quadratic differential system to have invariant straight lines with total multiplicity four (respectively five or six); ii) the complete list of phase portraits of all such systems; iii) the bifurcation diagram in the space $\mathbb{R}^{12}$ of the coefficients of the systems (see [108, 110–112]). Schlomiuk and Vulpe also gave necessary and sufficient affine invariant conditions for a quadratic system to be of Lotka-Volterra type. For these systems they also gave the full list of phase portraits as well as the bifurcation diagram of Lotka-Volterra systems in the space $\mathbb{R}^{12}$ of their coefficients.

Llibre and Vulpe classified all cubic systems possessing the maximum number 9 of invariant straight lines (real or complex) taking into account their multiplicities and the line at infinity [83]. They detected 23 topologically distinct configurations of invariant lines for such systems and proved that modulo the group $Aff(2,\mathbb{R})$ action and time rescaling, each configuration leads to just one point in the coefficient space $\mathbb{R}^{20}$. Moreover, every configuration is characterized by a set of affine invariant conditions. This classification was completed by the author of the Thesis Cristina Bujac in [16].

Vulpe in common with Bujac, have characterized in invariant terms the cubic differential systems having invariant straight lines of total multiplicity 8, including the line at infinity (see [17, 27–30]).

Baltag together with Calin proved that under generic assumptions the planar system $dx/dt = cx + dy + xC(x,y)$, $dy/dt = ex + fy + yC(x,y)$, where $R(x,y) = -ex^2 + (c-f)xy + dy^2 \neq 0$ and $C(x,y) \neq 0$ is a homogeneous polynomial of degree $r \geq 1$, has a first integral of generalized Darboux type. The main idea is to express an invariant algebraic curve of the system using comitants and invariants of the system (see [7]).

Baltag worked together with Vulpe to construct polynomial invariants helpful in the qualitative theory of polynomial differential systems. Some invariant polynomials with respect to affine transformations which they constructed in [10], [11] give affinely-invariant coefficient conditions for the total multiplicity of finite singularities for generic polynomial differential systems (see the polynomials μ_i, $i = 0, \ldots, n^2$ for $n = 3$ in Section 2.1). In particular, for the quadratic differential systems with real coefficients they found the conditions for the number and multiplicity of finite singular points (see [8], [9]). These invariant polynomials were used in many works in classification problems of quadratic or cubic differential systems such as for example [17, 28–30].

In another paper [9] Baltag and Vulpe provide a classification of planar quadratic differential systems, in terms of the number and multiplicity of the critical points on the line at infinity. The results of these papers are essentially based on the classification of the family of quadratic systems according to the total multiplicity of all finite singular points of systems.

Mihail Popa and his students (A. Braicov, O. Diaconescu, N. Gherstega, Anca-Veronica, P. Makari, E. Naidenova, V. Orlov, S. Port, V. Pricop) studied the finite-dimensional graduate algebras of comitants and invariants which he called *the Sibirsky algebras* [99], and they obtained characteristics of these algebras (the Krull dimension and the type and number of generators for autonomous polynomial differential systems, Hilbert series for the graded algebras of comitants and invariants of two-dimensional autonomous polynomial differential systems) (see, for instance [98]). They obtained invariant classifications of $GL(2, \mathbb{R})$ and $Aff(2, \mathbb{R})$-orbits for some classes of polynomial differential systems.

Popa and Pricop obtained an estimate for the number of algebraically independent Poincaré-Lyapunov constants for polynomial vector fields possessing a singular point of center or focus type (see [99]).

This theory of invariant polynomials also allowed us to classify topologically some classes of systems which have algebraic-geometric properties such as the Lotka-Volterra differential systems (see, for instance [116]).

The work of C. S. Sibirsky and the Chişinău school has implications which extend to modern, geometric invariant theory. For example, in the cubic case we need to classify the set $\mathbb{CS}$ of cubic differential systems according to their configurations of invariant lines. This set modulo the group action of the group $G = Aff(2, \mathbb{R}) \times \mathbb{R}^*$ of affine transformations and time rescaling is 13-dimensional. This is a very large number of parameters. Up to now only 1-parameter families of $\mathbb{CS}$ have been classified according to configurations of invariant lines. In the first part of this classification of $\mathbb{CS}$ the invariant theory approach intervenes in gluing the various charts of this 13-dimensional topological space of orbits under the G-action on $\mathbb{CS}$. In particular this is what is done in this Thesis for the problem of geometrically classifying the configurations of invariant lines of systems in the class $\mathbb{CSL}_8$. Of course the invariants and comitants are helpful in *completely* classifying $\mathbb{CS}$ according to specific algebraic-geometric properties of the systems. In particular it has allowed us to *completely* classify these systems according to the number of distinct singular point at infinity. More precisely, we split the whole set $\mathbb{CSL}_8$ of cubic systems with invariant lines of total multiplicity eight (which have the non-degenerate infinity) in four subfamilies of such systems with either 4 $\mathbb{ISP}s$ or 3 $\mathbb{ISP}s$ or 2 $\mathbb{ISP}s$ or 1 $\mathbb{ISP}$. Surely that these subfamilie of systems are distinguished by affine invariant conditions.

All 51 configurations of invariant lines obtained in the Thesis for $\mathbb{CSL}_8$ are distinguished by means of 52 new invariant polynomials, constructed in this work, besides 20 earlier constructed in [83] (see Section 2.1).

1.3. Conclusions on Chapter 1

Chapter 1 contains a survey of the most important results related to the purpose and objectives of the Thesis, the directions of investigations.

Our work in the Thesis was partly motivated by the following hard problems concerning polynomial differential systems and we discuss about this in Chapter 1:

1. the problem of topologically classifying all phase portraits of polynomial systems of a given degree n;

2. the problem of algebraic integrability (stated by Poincaré).

The subject of the Thesis is the set $\mathbb{CSL}_8$ of all cubic systems with invariant straight lines of total multiplicity eight. One of the main objective of this work is *to detect all possible configurations of invariant straight lines for systems in* $\mathbb{CSL}_8$.
This classification could serve as a base for either a further total topological classification of this family of systems or determining of the first integrals (integrating factors) of corresponding systems applying the method of integration of Darboux. Our assumptions are based on some arguments examined in Chapter 1.

In order to examine some specifical geometrical properties of systems (1.1) authors, as a rule, apply affine transformations and time rescaling which keeps these proprieties as well as the degree of the systems. As a result there are obtained some canonical systems (normal forms) which depend on less number of parameters. For these canonical systems the necessary and sufficient conditions in terms of their parameters which characterize the corresponding geometrical properties are constructed. However it is clear that these conditions are related with the canonical systems and they are not valid for the initial systems. The next objective of the current Thesis is *to construct necessary and sufficient affine invariant conditions for the realization of each one of the detected configurations of systems in* $\mathbb{CSL}_8$. These conditions, constructed by means of the invariant theory, help us to establish a connection between the canonical and initial systems because the applied affine transformations theoretically exist but they are unknown.
The roots of the invariant theory of polynomial vector fields lie in the classical invariant theory. The idea to adapt to polynomial vector fields the concepts of classical invariant theory is due to C.S. Sibirsky, the founder of the Chişinău school. In Chapter 1 we also describe briefly classical invariant theory and explain how the ideas of this theory were used by Sibirsky and his school to built an analogous theory for polynomial vector fields.

The cases of integrable systems are rare, but as Arnold said in [1, p.405] "...these integrable cases allow us to collect a large amount of information about the motion in more important systems...".

As it could be observed from Preliminaries, besides the Invariant Theory we have used in the Thesis the Qualitative Theory of Dynamical Systems, the Algebraic Theory of Resultants and Subresultants as well as Poincaré compactification and Bifurcation Theory.

2. CUBIC SYSTEMS WITH INVARIANT LINES OF TOTAL MULTIPLICITY EIGHT AND FOUR DISTINCT INFINITE SINGULARITIES

2.1. Preliminaries

In this subsection we give the basic notions and results which we need in this paper.

Consider planar real differential cubic systems, i.e. systems of the form:

$$\begin{aligned} \dot{x} &= p_0 + p_1(x,y) + p_2(x,y) + p_3(x,y) \equiv P(x,y), \\ \dot{y} &= q_0 + q_1(x,y) + q_2(x,y) + q_3(x,y) \equiv Q(x,y) \end{aligned} \tag{2.1}$$

with real coefficients and variables x and y. The polynomials p_i and q_i $(i = 0, 1, 2, 3)$ are homogeneous polynomials of degree i in x and y:

$$\begin{aligned} &p_0 = a_{00}, \quad p_3(x,y) = a_{30}x^3 + 3a_{21}x^2y + 3a_{12}xy^2 + a_{03}y^3, \\ &p_1(x,y) = a_{10}x + a_{01}y, \quad p_2(x,y) = a_{20}x^2 + 2a_{11}xy + a_{02}y^2, \\ &q_0 = b_{00}, \quad q_3(x,y) = b_{30}x^3 + 3b_{21}x^2y + 3b_{12}xy^2 + b_{03}y^3, \\ &q_1(x,y) = b_{10}x + b_{01}y, \quad q_2(x,y) = b_{20}x^2 + 2b_{11}xy + b_{02}y^2. \end{aligned}$$

Let $a = (a_{00}, a_{10}, a_{01}, \dots, a_{03}, b_{00}, b_{10}, b_{01}, \dots, b_{03})$ be the 20-tuple of the coefficients of systems (2.1) and denote $\mathbb{R}[a,x,y] = \mathbb{R}[a_{00}, a_{10}, a_{01}, \dots, a_{03}, b_{00}, b_{10}, b_{01}, \dots, b_{03}, x, y]$.

The systems (2.1) we could write in the coefficient form:

$$(S) \quad \begin{cases} \dot{x} = & a + cx + dy + gx^2 + 2hxy + ky^2 + px^3 + 3qx^2y + 3rxy^2 + sy^3, \\ \dot{y} = & b + ex + fy + lx^2 + 2mxy + ny^2 + tx^3 + 3ux^2y + 3vxy^2 + wy^3. \end{cases} \tag{2.2}$$

2.1.1. Main invariant polynomials associated to configurations of invariant lines

It is known that on the set $\mathbb{CS}$ of all cubic differential systems (2.1) acts the group $Aff(2,\mathbb{R})$ of affine transformations on the plane [108]. Indeed for every $g \in Aff(2,\mathbb{R})$, $g : \mathbb{R}^2 \to \mathbb{R}^2$ we have:

$$g : \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = M \begin{pmatrix} x \\ y \end{pmatrix} + B; \quad g^{-1} : \begin{pmatrix} x \\ y \end{pmatrix} = M^{-1} \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} - M^{-1}B$$

where $M = \|M_{ij}\|$ is a 2×2 nonsingular matrix and B is a 2×1 matrix over $\mathbb{R}$. For every $S \in \mathbb{CS}$ we can form its transformed system $\bar{S} = g \cdot S$:

$$d\bar{x}/dt = \bar{P}(\bar{x},\bar{y}), \quad d\bar{y}/dt = \bar{Q}(\bar{x},\bar{y})$$

where $\left(\bar{P}(\bar{x},\bar{y}) \quad \bar{Q}(\bar{x},\bar{y})\right)^t = M\left((P \circ g^{-1})(\bar{x},\bar{y}) \quad (Q \circ g^{-1})(\bar{x},\bar{y})\right)^t$.

For every $g \in Aff(2,\mathbb{R})$ let $r_g : \mathbb{R}^{20} \to \mathbb{R}^{20}$ be the map which corresponds to g via this action. We know (see [122]) that r_g is linear and that the map $r : Aff(2,\mathbb{R}) \to GL(20,\mathbb{R})$ thus obtained is a group homomorphism. For every subgroup $G \subseteq Aff(2,\mathbb{R})$, r induces a representation of G onto a subgroup $\mathcal{G}$ of $GL(20,\mathbb{R})$.

Next we need the following definitions which were used in [108] (see also [122]).

Definition 2.1 ([108]). *A polynomial $U(a,x,y) \in \mathbb{R}[a,x,y]$ is called a comitant of systems* (2.1) *with respect to a subgroup $G \subseteq Aff(2,\mathbb{R})$ if there exists $\chi \in \mathbb{Z}$ such that for every $(g,a) \in G \times \mathbb{R}^{20}$ and for every $(x,y) \in \mathbb{R}^2$ the following relation holds:*

$$U(r_g(a), g(x,y)) \equiv (detg)^{-\chi} U(a,x,y),$$

where $\det g = \det M$. *If the polynomial U does not exactly depend on x and y then it is called invariant. The number $\chi \in \mathbb{Z}$ is called the weight of the comitant $U(a,x,y)$. If $G = GL(2,\mathbb{R})$ (or $G = Aff(2,\mathbb{R})$) then the comitant $U(a,x,y)$ of systems* (2.1) *is called $GL-$comitant (respectively, affine comitant).*

Definition 2.2 ([108]). *A subset $X \subset \mathbb{R}^{20}$ will be called $G-$invariant, if for every $g \in G$ we have $r_g(X) \subseteq X$.*

Let us consider the polynomials

$$\begin{aligned} C_i(a,x,y) &= yp_i(a,x,y) - xq_i(a,x,y) \in \mathbb{R}[a,x,y],\ i = 0,1,2,3, \\ D_i(a,x,y) &= \frac{\partial}{\partial x}p_i(a,x,y) + \frac{\partial}{\partial y}q_i(a,x,y) \in \mathbb{R}[a,x,y],\ i = 1,2,3. \end{aligned} \tag{2.3}$$

As it was shown in [122], the polynomials $\{C_0(a,x,y),\ C_1(a,x,y),\ C_2(a,x,y),\ C_3(a,x,y),\ D_1(a),\ D_2(a,x,y)\ D_3(a,x,y)\}$ of degree one in the coefficients of systems (2.1) are GL-comitants of these systems.

Let $T(2,\mathbb{R})$ be the subgroup of $Aff(2,\mathbb{R})$ formed by translations. For every $\tau \in T(2,\mathbb{R})$ such that $\tau : x = \bar{x} + \alpha,\ y = \bar{y} + \beta$ we consider $r_\tau : \mathbb{R}^{20} \to \mathbb{R}^{20}$.

Definition 2.3 ([108]). *A $GL-$comitant $U(a,x,y)$ of systems* (2.1) *is called a $T-$comitant if for every $(\tau,a) \in T(2,\mathbb{R}) \times \mathbb{R}^{20}$ and for every $(\bar{x},\bar{y}) \in \mathbb{R}^2$ the relation $U(r_\tau \cdot a, \bar{x}, \bar{y}) = U(a,\bar{x},\bar{y})$ holds.*

Let $U_i(a,x,y) = \sum_{j=0}^{d_i} U_{ij}(a) x^{d_i-j} y^j,\ i = 1,\dots,s$ be a set of $GL-$comitants of systems (2.1) where d_i denotes the degree of the binary form $U_i(a,x,y)$ in x and y. Additionally we denote by $\mathcal{U} = \{U_{ij}(a) \in \mathbb{R}[a] \mid i = 1,\dots,s,\ j = 0,\dots,d_i\}$ the set of the coefficients in $\mathbb{R}[a]$ of the $GL-$comitants $U_i(a,x,y),\ i = 1,\dots,s$ and by $V(\mathcal{U})$ its associated algebraic set: $V(\mathcal{U}) = \{a \in \mathbb{R}^{20} \mid U_{ij}(a) = 0 \quad \text{for every} \quad U_{ij}(a) \in \mathcal{U}\}$.

Definition 2.4 ([34, 108]). *A GL−comitant $U(a, x, y)$ of systems (2.1) is called a conditional T−comitant (CT−comitant) modulo $< U_1, \dots, U_s >$ if the following two conditions are satisfied:*

(i) the algebraic subset $V(\mathcal{U}) \subset \mathbb{R}^{20}$ is affine invariant;

(ii) for every $(\tau, a) \in T(2, \mathbb{R}) \times V(\mathcal{U})$ we have $U(r_\tau \cdot a, \bar{x}, \bar{y}) = U(a, \bar{x}, \bar{y})$ in $\mathbb{R}[\bar{x}, \bar{y}]$.

Notation 2.1. *Let $f,\ g \in \mathbb{R}[a, x, y]$ and*

$$(f, g)^{(k)} = \sum_{h=0}^{k} (-1)^h \binom{k}{h} \frac{\partial^k f}{\partial x^{k-h} \partial y^h} \frac{\partial^k g}{\partial x^h \partial y^{k-h}}. \tag{2.4}$$

$(f, g)^{(k)} \in \mathbb{R}[a, x, y]$ is called the transvectant of index k of (f, g) (see for instance [68], [87]).

Theorem 2.1 ([136]). *Any GL-comitant of systems (2.1) can be constructed from the elements of the set $\{C_i,\ D_i,\ i = 0, 1, 2, 3\}$ by using the operations: $+$, $-$, $\times$, and by applying the differential operation $(f, g)^{(k)}$.*

In order to define the needed invariant polynomials we first construct the following comitants of second degree with respect to the coefficients of the initial system:

$$\begin{array}{lll}
S_1 = (C_0, C_1)^{(1)}, & S_{10} = (C_1, C_3)^{(1)}, & S_{19} = (C_2, D_3)^{(1)}, \\
S_2 = (C_0, C_2)^{(1)}, & S_{11} = (C_1, C_3)^{(2)}, & S_{20} = (C_2, D_3)^{(2)}, \\
S_3 = (C_0, D_2)^{(1)}, & S_{12} = (C_1, D_3)^{(1)}, & S_{21} = (D_2, C_3)^{(1)}, \\
S_4 = (C_0, C_3)^{(1)}, & S_{13} = (C_1, D_3)^{(2)}, & S_{22} = (D_2, D_3)^{(1)}, \\
S_5 = (C_0, D_3)^{(1)}, & S_{14} = (C_2, C_2)^{(2)}, & S_{23} = (C_3, C_3)^{(2)}, \\
S_6 = (C_1, C_1)^{(2)}, & S_{15} = (C_2, D_2)^{(1)}, & S_{24} = (C_3, C_3)^{(4)}, \\
S_7 = (C_1, C_2)^{(1)}, & S_{16} = (C_2, C_3)^{(1)}, & S_{25} = (C_3, D_3)^{(1)}, \\
S_8 = (C_1, C_2)^{(2)}, & S_{17} = (C_2, C_3)^{(2)}, & S_{26} = (C_3, D_3)^{(2)}, \\
S_9 = (C_1, D_2)^{(1)}, & S_{18} = (C_2, C_3)^{(3)}, & S_{27} = (D_3, D_3)^{(2)}.
\end{array}$$

We shall use here the following invariant polynomials constructed in [83] to characterize the family of cubic systems possessing the maximal number (i.e. nine) of invariant straight lines:

$$\begin{array}{rcl}
\mathcal{D}_1(a) &=& 6S_{24}^3 - \left[(C_3, S_{23})^{(4)}\right]^2, \quad \mathcal{D}_2(a, x, y) = -S_{23}, \\
\mathcal{D}_3(a, x, y) &=& (S_{23},\ S_{23})^{(2)} - 6C_3(C_3,\ S_{23})^{(4)}, \quad \mathcal{D}_4(a) = (C_3,\ \mathcal{D}_2)^{(4)}, \\
\mathcal{V}_1(a, x, y) &=& S_{23} + 2D_3^2, \quad \mathcal{V}_2(a, x, y) = S_{26}, \quad \mathcal{V}_3(a, x, y) = 6S_{25} - 3S_{23} - 2D_3^2, \\
\mathcal{V}_4(a, x, y) &=& C_3\left[(C_3, S_{23})^{(4)} + 36\,(D_3, S_{26})^{(2)}\right],
\end{array}$$

$$\begin{aligned}
\mathcal{L}_1(a,x,y) &= 9C_2\left(S_{24}+24S_{27}\right)-12D_3\left(S_{20}+8S_{22}\right)-12\left(S_{16},D_3\right)^{(2)}-3\left(S_{23},C_2\right)^{(2)}-\\
&\quad -16\left(S_{19},C_3\right)^{(2)}+12\left(5S_{20}+24S_{22},C_3\right)^{(1)},\\
\mathcal{L}_2(a,x,y) &= 32\left(13S_{19}+33S_{21},D_2\right)^{(1)}+84\left(9S_{11}-2S_{14},D_3\right)^{(1)}-448\left(S_{18},C_2\right)^{(1)}+\\
&\quad +8D_2\left(12S_{22}+35S_{18}-73S_{20}\right)-56\left(S_{17},C_2\right)^{(2)}-63\left(S_{23},C_1\right)^{(2)}+\\
&\quad +756D_3S_{13}-1944D_1S_{26}+112\left(S_{17},D_2\right)^{(1)}-378\left(S_{26},C_1\right)^{(1)}+\\
&\quad +9C_1\left(48S_{27}-35S_{24}\right),\quad \mathcal{U}_1(a)=S_{24}-4S_{27},\\
\mathcal{U}_2(a,x,y) &= 6(S_{23}-3S_{25},S_{26})^{(1)}-3S_{23}(S_{24}-8S_{27})-24S_{26}^2+2C_3(C_3,S_{23})^{(4)}+\\
&\quad +24D_3(D_3,S_{26})^{(1)}+24D_3^2S_{27}.
\end{aligned}$$

However these invariant polynomials are not sufficient to characterize the cubic systems with invariant lines of total multiplicity 8. So we construct here the following new invariant polynomials:

$$\begin{aligned}
\mathcal{V}_5(a,x,y) &= 6T_1(9A_5-7A_6)+2T_2(4T_{16}-T_{17})-3T_3(3A_1+5A_2)+3A_2T_4+36T_5^2-3T_{44},\\
\mathcal{V}_6(a,x,y) &= 6D_3^2+S_{23}+6S_{25},\quad \mathcal{L}_6(a)=2A_3-19A_4,\quad \mathcal{L}_7(a,x,y)=(T_{10},T_{10})^{(2)},\\
\mathcal{K}_1(a,x,y) &= \left(3223T_2^2T_{140}+2718T_4T_{140}-829T_2^2T_{141},T_{133}\right)^{(10)}/2,\quad \mathcal{K}_2(a,x,y)=T_{74},\\
\mathcal{K}_3(a,x,y) &= Z_1Z_2Z_3,\quad \mathcal{K}_4(a,x,y)=T_{13}-2T_{11},\\
\mathcal{K}_5(a,x,y) &= 45T_{42}-T_2T_{14}+2T_2T_{15}+12T_{36}+45T_{37}-45T_{38}+30T_{39},\\
\mathcal{K}_6(a,x,y) &= 4T_1T_8(2663T_{14}-8161T_{15})+6T_8(178T_{23}+70T_{24}+555T_{26})+\\
&\quad 18T_9(30T_2T_8-488T_1T_{11}-119T_{21})+5T_2(25T_{136}+16T_{137})-\\
&\quad -15T_1(25T_{140}-11T_{141})-165T_{142},\\
\mathcal{K}_7(a) &= A_1+3A_2,\quad \mathcal{K}_8(a,x,y)=10A_4T_1-3T_2T_{15}+4T_{36}-8T_{37},\\
\mathcal{K}_9(a,x,y) &= 3T_1(11T_{15}-8T_{14})-T_{23}+5T_{24},\\
N_1(a,x,y) &= S_{13},\quad N_2(a,x,y)=T_9,\quad N_3(a,x,y)=C_2D_3+3S_{16},\\
N_4(a,x,y) &= -S_{14}^2-2D_2^2(3S_{14}-8S_{15})-12D_3(S_{14},C_1)^{(1)}+\\
&\quad +D_2(-48D_3S_9+16(S_{17},C_1)^{(1)}),\\
N_5(a,x,y) &= 36D_2D_3(S_8-S_9)+D_1(108D_2^2D_3-54D_3(S_{14}-8S_{15}))+\\
&\quad +2S_{14}(S_{14}-22S_{15})-8D_2^2(3S_{14}+S_{15})-9D_3(S_{14},C_1)^{(1)}-16D_2^4,\\
N_6(a,x,y) &= 40D_3^2(15S_6-4S_3)-480D_2D_3S_9-20D_1D_3(S_{14}-4S_{15})+\\
&\quad +160D_2^2S_{15}-35D_3(S_{14},C_1)^{(1)}+8\left((S_{23},C_2)^{(1)},C_0\right)^{(1)},
\end{aligned}$$

$$
\begin{aligned}
N_7(a,x,y) &= 18C_2D_2(9D_1D_3-S_{14})-2C_1D_3(8D_2^2-3S_{14}-74S_{15})-\\
&\quad -432C_0D_3S_{21}48S_7(8D_2D_3+S_{17})+6S_{10}(12D_2^2+151S_{15})-\\
&\quad -51S_{10}S_{14}-162D_1D_2S_{16}+864D_3(S_{16},C_0)^{(1)},\\
N_8(a,x,y) &= -32D_3^2S_2-108D_1D_3S_{10}+108C_3D_1S_{11}-18C_1D_3S_{11}-27S_{10}S_{11}+\\
&\quad +4C_0D_3(9D_2D_3+4S_{17})+108S_4S_{21},\\
N_9(a,x,y) &= 11S_{14}^2-16D_1D_3(16D_2^2+19S_{14}-152S_{15})-8D_2^2(7S_{14}+32S_{15})-\\
&\quad -2592D_1^2S_{25}+88D_2(S_{14},C_2)^{(1)},\\
N_{10}(a,x,y) &= -24D_1D_3+4D_2^2+S_{14}-8S_{15},\\
N_{11}(a,x,y) &= S_{14}^2+D_1[16D_2^2D_3-8D_3(S_{14}-8S_{15})]-2D_2^2(5S_{14}-8S_{15})+\\
&\quad +8D_2(S_{14},C_2)^{(1)},\\
N_{12}(a,x,y) &= -160D_2^4-1620D_3^2S_3+D_1(1080D_2^2D_3-135D_3(S_{14}-20S_{15}))-\\
&\quad -5D_2^2(39S_{14}-32S_{15})+85D_2(S_{14},C_2)^{(1)}+81\big((S_{23},C_2)^{(1)},C_0\big)^{(1)}+5S_{14}^2,\\
N_{13}(a,x,y) &= 2(136D_3^2S_2-126D_2D_3S_4+60D_2D_3S_7+63S_{10}S_{11})-\\
&\quad -18C_3D_1(S_{14}-28S_{15})-12C_1D_3(7S_{11}-20S_{15})-192C_2D_2S_{15}+\\
&\quad +4C_0D_3(21D_2D_3+17S_{17})+3C_2(S_{14},C_2)^{(1)},\\
N_{14}(a,x,y) &= -6D_1D_3-15S_{12}+2S_{14}+4S_{15},\\
N_{15}(a,x,y) &= 216D_1D_3(63S_{11}-104D_2^2-136S_{15})+4536D_3^2S_6+4096D_2^4+\\
&\quad +120S_{14}^2+992D_2(S_{14},C_2)^{(1)}-135D_3\big[28(S_{17},C_0)^{(1)}+5(S_{14},C_1)^{(1)}\big],\\
N_{16}(a,x,y) &= 2C_1D_3+3S_{10},\ \ N_{17}(a,x,y)=6D_1D_3-2D_2^2-(C_3,C_1)^{(2)},\\
N_{18}(a,x,y) &= 2D_2^3-6D_1D_2D_3-12D_3S_5+3D_3S_8,\\
N_{19}(a,x,y) &= C_1D_3(18D_1^2-S_6)+C_0(4D_2^3-12D_1D_2D_3-18D_3S_5+9D_3S_8)+6C_2D_1S_8+\\
&\quad +2\big(9D_2D_3S_1-4D_2^2S_2+12D_1D_3S_2-9C_3D_1S_6-9D_3(S_4,C_0)^{(1)}\big),\\
N_{20}(a,x,y) &= 3D_2^4-8D_1D_2^2D_3-8D_3^2S_6-16D_1D_3S_{11}+16D_2D_3S_9,\\
N_{21}(a,x,y) &= 2D_1D_2^2D_3-4D_3^2S_6+D_2D_3S_8+D_1(S_{23},C_1)^{(1)},\\
N_{22}(a,x,y) &= T_8,\quad N_{23}(a,x,y)=T_6,\quad N_{24}(a,x,y)=2T_3T_{74}-T_1T_{136},\\
N_{25}(a,x,y) &= 5T_3T_6-T_1T_{23},\ \ N_{26}=9T_{135}-480T_6T_8-40T_2T_{74}-15T_2T_{75},\\
N_{27}(a,x,y) &= 9T_2T_9(2T_{23}-5T_{24}-80T_{25})+144T_{25}(T_{23}+5T_{24}+15T_{26})-\\
&\quad -9(T_{23}^2-5T_{24}^2-33T_9T_{76}),\quad N_{28}(a,x,y)=T_3+T_4,
\end{aligned}
$$

$$
\begin{aligned}
W_1(a,x,y) &= 2C_2D_3-3C_3D_2,\\
W_2(a,x,y) &= 6C_3(S_{12}+6S_{11})-9C_1(S_{23}+S_{25})-8(S_{16},C_2)^{(1)}-C_3D_2^2,\\
W_3(a,x,y) &= 12D_1C_3-S_{10},\ W_4(a,x,y)=-27S_4+4S_7,\\
W_5(a,x,y) &= 3D_1^2C_1+4D_1S_2-3(S_4,C_0)^{(1)},\\
W_6(a,x,y) &= 2C_2D_1+3S_4,\ W_7(a,x,y)=(S_{10},D_2)^{(1)},\\
W_8(a,x,y) &= 4C_2(27D_1D_3-8D_2^2)+2C_2(20S_{15}-4S_{14}+39S_{12})+18C_1(3S_{21}-D_2D_3)+\\
&\quad +54D_3(3S_4-S_7)-288C_3S_9+54(S_7,C_3)^{(1)}-567(S_4,C_3)^{(1)}+135C_0D_3^2,\\
W_9(a,x,y) &= 3S_6D_2^2+4S_3D_2^2-6D_1D_2S_9,\\
W_{10}(a,x,y) &= 18D_1^2C_2+15S_6C_2-6D_1C_1D_2+4C_0D_2^2+27D_1S_4-6C_1S_9,\\
W_{11}(a,x,y) &= 9C_0D_3^5-6D_3^4(C_1D_2-S_7)+4C_2D_3^3(D_2^2+S_{14}-2S_{15})-\\
&\quad -12C_3D_3^2[5D_2S_{14}-4D_2S_{15}-7(S_{14},C_2)^{(1)}],\\
W_{12}(a,x,y) &= -480T_6T_8+9T_{135}-40T_2T_{74}-15T_2T_{75}
\end{aligned}
$$

where

$$
\begin{aligned}
Z_1 &= 2C_1D_2D_3-9C_0(S_{25}+2D_3^2)+4C_2(9D_1D_3+S_{14})-3C_3(6D_1D_2+5S_8)+36D_3S_4,\\
Z_2 &= 12D_1S_{17}+2D_2(3S_{11}-2S_{14})+6D_3(S_8-6S_5)-9(S_{25},C_0)^{(1)},\\
Z_3 &= 48D_1^3C_3+12D_1^2(C_1D_3-C_2D_2)+36D_1(C_0S_{17}-C_3S_6)-16D_2^2S_2-16S_2S_{14}+\\
&\quad +2C_0D_2(3S_{11}+2S_{14})+3D_3(8D_2S_1+3C_0S_8-2C_1S_6)-9S_4S_8\\
&\quad -216C_3(S_5,C_0)^{(1)})+6C_2(D_2S_6-4(S_{14},C_0)^{(1)})+54D_1D_2(S_4+D_3C_0).
\end{aligned}
$$

Here the polynomials

$$
\begin{aligned}
A_1 &= S_{24}/288,\quad A_2=S_{27}/72,\ A_3=(72D_1A_2+\big(S_{22},D_2\big)^{(1)})/24,\\
A_4 &= \big[9D_1(S_{24}-288A_2)+4\big(9S_{11}-2S_{14},D_3\big)^{(2)}+8\big(3S_{18}-S_{20}-4S_{22},D_2\big)^{(1)}\big]/2^7/3^3,\\
A_5 &= \big(S_{23},C_3\big)^{(4)}/2^7/3^5,\quad A_6=\big(S_{26},D_3\big)^{(2)}/2^5/3^3
\end{aligned}
$$

are affine invariants, whereas the polynomials

$$
\begin{aligned}
T_1 &= C_3,\quad T_2= D_3,\quad T_3= S_{23}/18,\quad T_4= S_{25}/6,\quad T_5= S_{26}/72,\\
T_6 &= \big[3C_1(D_3^2-9T_3+18T_4)-2C_2(2D_2D_3-S_{17}+2S_{19}-6S_{21})+\\
&\quad +2C_3(2D_2^2-S_{14}+8S_{15})\big]/2^4/3^2,\\
T_8 &= \big[5D_2(D_3^2+27T_3-18T_4)+20D_3S_{19}+12\big(S_{16},D_3\big)^{(1)}-8D_3S_{17}\big]/5/2^5/3^3,
\end{aligned}
$$

$$T_9 = \big[9D_1(9T_3-18T_4-D_3^2)+2D_2(D_2D_3-3S_{17}-S_{19}-9S_{21})+18\big(S_{15},C_3\big)^{(1)}-$$
$$-6C_2(2S_{20}-3S_{22})+18C_1S_{26}+2D_3S_{14}\big]/2^4/3^3,$$
$$T_{11} = \big[\big(D_3^2-9T_3+18T_4,C_2\big)^{(2)}-6\big(D_3^2-9T_3+18T_4,D_2\big)^{(1)}-12\big(S_{26},C_2\big)^{(1)}+$$
$$+12D_2S_{26}+432(A_1-5A_2)C_2\big]/2^7/3^4,$$
$$T_{13} = \big[27(T_3,C_2)^{(2)}-18(T_4,C_2)^{(2)}+48D_3S_{22}-216(T_4,D_2)^{(1)}+36D_2S_{26}-$$
$$-1296C_2A_1-7344C_2A_2+(D_3^2,C_2)^{(2)}\big]/2^7/3^4,$$
$$T_{14} = \big[\big(8S_{19}+9S_{21},D_2\big)^{(1)}-D_2(8S_{20}+3S_{22})+18D_1S_{26}+1296C_1A_2\big]/2^4/3^3,$$
$$T_{15} = 8\big(9S_{19}+2S_{21},D_2\big)^{(1)}+3\big(9T_3-18T_4-D_3^2,C_1\big)^{(2)}-4\big(S_{17},C_2\big)^{(2)}+$$
$$+4\big(S_{14}-17S_{15},D_3\big)^{(1)}-8\big(S_{14}+S_{15},C_3\big)^{(2)}+432C_1(5A_1+11A_2)+$$
$$+36D_1S_{26}-4D_2(S_{18}+4S_{22})\big]/2^6/3^3,$$
$$T_{21} = \big(T_8,C_3\big)^{(1)},\quad T_{23}=\big(T_6,C_3\big)^{(2)}/6,\quad T_{24}=\big(T_6,D_3\big)^{(1)}/6,$$
$$T_{26} = \big(T_9,C_3\big)^{(1)}/4,\quad T_{30}=\big(T_{11},C_3\big)^{(1)},\quad T_{31}=\big(T_8,C_3\big)^{(2)}/24,$$
$$T_{32} = \big(T_8,D_3\big)^{(1)}/6,\quad T_{36}=\big(T_6,D_3\big)^{(2)}/12,\quad T_{37}=\big(T_9,C_3\big)^{(2)}/12,$$
$$T_{38} = \big(T_9,D_3\big)^{(1)}/12,\quad T_{39}=\big(T_6,C_3\big)^{(3)}/2^4/3^2,\quad T_{42}=\big(T_{14},C_3\big)^{(1)}/2,$$
$$T_{44} = \big((S_{23},C_3)^{(1)},D_3\big)^{(2)}/5/2^6/3^3,$$
$$T_{74} = \big[27C_0(9T_3-18T_4-D_3^2)^2+C_1\big(-62208T_{11}C_3-3(9T_3-18T_4-D_3^2)\times$$
$$\times(2D_2D_3-S_{17}+2S_{19}-6S_{21})\big)+20736T_{11}C_2^2+C_2(9T_3-18T_4-D_3^2)\times$$
$$\times(8D_2^2+54D_1D_3-27S_{11}+27S_{12}-4S_{14}+32S_{15})-54C_3(9T_3-18T_4-D_3^2)\times$$
$$\times(2D_1D_2-S_8+2S_9)-54D_1(9T_3-18T_4-D_3^2)S_{16}-$$
$$-576T_6(2D_2D_3-S_{17}+2S_{19}-6S_{21})\big]/2^8/3^4,\quad T_{133}=(T_{74},C_3)^{(1)},$$
$$T_{136} = \big(T_{74},C_3\big)^{(2)}/24,\quad T_{137}=\big(T_{74},D_3\big)^{(1)}/6,\quad T_{140}=\big(T_{74},D_3\big)^{(2)}/12,$$
$$T_{141} = \big(T_{74},C_3\big)^{(3)}/36,\quad T_{142}=\big((T_{74},C_3)^{(2)},C_3\big)^{(1)}/72$$

are T-comitants of cubic systems (2.1). We note that these polynomials are the elements of the polynomial basis of T-comitants up to degree six constructed by Iu. Calin [37].

Next we consider the differential operator $\mathcal{L}=x\cdot\mathbf{L}_2-y\cdot\mathbf{L}_1$ constructed in [10] and acting on $\mathbb{R}[a,x,y]$, where

$$\mathbf{L}_1 = 3a_{00}\frac{\partial}{\partial a_{10}}+2a_{10}\frac{\partial}{\partial a_{20}}+a_{01}\frac{\partial}{\partial a_{11}}+\frac{1}{3}a_{02}\frac{\partial}{\partial a_{12}}+\frac{2}{3}a_{11}\frac{\partial}{\partial a_{21}}+a_{20}\frac{\partial}{\partial a_{30}}+$$
$$3b_{00}\frac{\partial}{\partial b_{10}}+2b_{10}\frac{\partial}{\partial b_{20}}+b_{01}\frac{\partial}{\partial b_{11}}+\frac{1}{3}b_{02}\frac{\partial}{\partial b_{12}}+\frac{2}{3}b_{11}\frac{\partial}{\partial b_{21}}+b_{20}\frac{\partial}{\partial b_{30}},$$

$$\mathbf{L}_2 = 3a_{00}\frac{\partial}{\partial a_{01}} + 2a_{01}\frac{\partial}{\partial a_{02}} + a_{10}\frac{\partial}{\partial a_{11}} + \frac{1}{3}a_{20}\frac{\partial}{\partial a_{21}} + \frac{2}{3}a_{11}\frac{\partial}{\partial a_{12}} + a_{02}\frac{\partial}{\partial a_{03}} +$$
$$3b_{00}\frac{\partial}{\partial b_{01}} + 2b_{01}\frac{\partial}{\partial b_{02}} + b_{10}\frac{\partial}{\partial b_{11}} + \frac{1}{3}b_{20}\frac{\partial}{\partial b_{21}} + \frac{2}{3}b_{11}\frac{\partial}{\partial b_{12}} + b_{02}\frac{\partial}{\partial b_{03}}.$$

Using this operator and the affine invariant $\mu_0 = \text{Resultant}_x\big(p_3(a,x,y), q_3(a,x,y)\big)/y^9$ we construct the following polynomials: $\mu_i(a,x,y) = \frac{1}{i!}\mathcal{L}^{(i)}(\mu_0)$, $i = 1,..,9$, where $\mathcal{L}^{(i)}(\mu_0) = \mathcal{L}(\mathcal{L}^{(i-1)}(\mu_0))$ and $\mathcal{L}^{(0)}(\mu_0) = \mu_0$.

These polynomials are in fact comitants of systems (2.1) with respect to the group $GL(2,\mathbb{R})$ (see [10]). The polynomial $\mu_i(a,x,y)$, $i \in \{0,1,\ldots,9\}$ is homogeneous of degree 6 in the coefficients of systems (2.1) and homogeneous of degree i in the variables x and y. The geometrical meaning of these polynomial is revealed in the next lemma.

Lemma 2.1 (see [6], [10])**.** *Assume that a cubic system (S) with coefficients $\tilde{a}$ belongs to the family* (2.1)*. Then:*

(i) The total multiplicity of all finite singularities of this system equals $9-k$ if and only if for every $i \in \{0,1,\ldots,k-1\}$ we have $\mu_i(\tilde{a},x,y) = 0$ in the ring $\mathbb{R}[x,y]$ and $\mu_k(\tilde{a},x,y) \neq 0$. In this case the factorization $\mu_k(\tilde{a},x,y) = \prod_{i=1}^{k}(u_i x - v_i y) \neq 0$ over $\mathbb{C}$ indicates the coordinates $[v_i : u_i : 0]$ of those finite singularities of the system (S) which "have gone" to infinity. Moreover the number of distinct factors in this factorization is less than or equal to four (the maximum number of infinite singularities of a cubic system) and the multiplicity of each one of the factors $u_i x - v_i y$ gives us the number of the finite singularities of the system (S) which have collapsed with the infinite singular point $[v_i : u_i : 0]$.

(ii) The system (S) is degenerate (i.e. $\gcd(P,Q) \neq \text{const}$) if and only if $\mu_i(\tilde{a},x,y) = 0$ in $\mathbb{R}[x,y]$ for every $i = 0,1,\ldots,9$.

The study of cubic systems when the total multiplicity of invariant straight lines (including the line at infinity) equals nine was done in [83]. For this propose in [83] there are proved some lemmas concerning the number of triplets and/or couples of parallel invariant straight lines which could have a cubic system. In [27] we complete these results proving the following theorem.

Theorem 2.2. *If a cubic system* (2.1) *possesses a given number of triplets or/and couples of invariant parallel affine lines real or/and complex, then the following conditions are satisfied, respectively:*

(i)	*2 triplets*	$\Rightarrow$	$\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{U}_1 = 0$;
(ii)	*1 triplet and 2 couples*	$\Rightarrow$	$\mathcal{V}_3 = \mathcal{V}_4 = \mathcal{U}_2 = 0$;
(iii)	*1 triplet and 1 couple*	$\Rightarrow$	$\mathcal{V}_4 = \mathcal{V}_5 = \mathcal{U}_2 = 0$;
(iv)	*1 triplet*	$\Rightarrow$	$\mathcal{V}_4 = \mathcal{U}_2 = 0$;
(v)	*3 couples*	$\Rightarrow$	$\mathcal{V}_3 = 0$;
(vi)	*2 couples*	$\Rightarrow$	$\mathcal{V}_5 = 0$.

Infinite singular points and associated homogeneous cubic canonical systems. It is well known that the infinite singularities (real or/and complex) of systems (2.1) are determined by the linear factors of the polynomial $C_3(x, y) = yp_3(x, y) - xq(x, y)$. According to [95] (see also [58]) we have the following result.

Lemma 2.2. *The number of distinct factors (real and imaginary) of the polynomial $C_3 \neq 0$ is determined by the following conditions:*

[i] 4 *real if* $\mathcal{D}_1 > 0$, $\mathcal{D}_2 > 0$, $\mathcal{D}_3 > 0$;

[ii] 2 *real and* 2 *imaginary if* $\mathcal{D}_1 < 0$;

[iii] 4 *imaginary if* $\mathcal{D}_1 > 0$ *and for every* (x, y) *where* $\mathcal{D}_2\mathcal{D}_3 \neq 0$ *either* $\mathcal{D}_2 < 0$ *or* $\mathcal{D}_3 < 0$;

[iv] 3 *real (*1 *double,* 2 *simple) if* $\mathcal{D}_1 = 0$, $\mathcal{D}_3 > 0$;

[v] 1 *real and* 2 *imaginary (*1 *real double) if* $\mathcal{D}_1 = 0$, $\mathcal{D}_3 < 0$;

[vi] 2 *real (*1 *triple and* 1 *simple) if* $\mathcal{D}_1 = \mathcal{D}_3 = 0$, $\mathcal{D}_2 \neq 0$, $\mathcal{D}_4 = 0$;

[vii] 2 *real (*2 *double) if* $\mathcal{D}_1 = \mathcal{D}_3 = 0$, $\mathcal{D}_2 > 0$, $\mathcal{D}_4 \neq 0$;

[viii] 2 *imaginary (*2 *double) if* $\mathcal{D}_1 = \mathcal{D}_3 = 0$, $\mathcal{D}_2 < 0$, $\mathcal{D}_4 \neq 0$;

[ix] 1 *real (of the multiplicity* 4*) if* $\mathcal{D}_1 = \mathcal{D}_2 = \mathcal{D}_3 = 0$

where $\mathcal{D}_i$ *for* $i = 1, 2, 3, 4$ *are the* $T-$ *comitants earlier defined in this subsection.*

We consider the polynomial $C_3(a, x, y) \neq 0$ as a quartic binary form. Here $a \in \mathbb{C}$ is *imaginary* if $a \notin \mathbb{R}$. It is well known that there exists $g \in GL(2, \mathbb{R})$, $g(x, y) = (u, v)$, such that the transformed binary form $gC_3(a, x, y) = C_3(a, g^{-1}(u, v))$ is one of the following 9 canonical forms:

(i)	$xy(x-y)(rx+sy),\ rs(r+s) \neq 0$;	(iv)	$x^2y(x-y)$;	(vii)	x^2y^2;
(ii)	$x(sx+y)(x^2+y^2)$;	(v)	$x^2(x^2+y^2)$;	$(viii)$	$(x^2+y^2)^2$;
(iii)	$(px^2+qy^2)(x^2+y^2),\ pq > 0$;	(vi)	x^3y;	(ix)	x^4.

We note that each one of the above canonical forms corresponds to one of the cases enumerated in the statement of Lemma 2.2.

On the other hand, applying the same transformation g to the initial system and calculating for the transformed system its polynomial $C_3(a(g), u, v)$ the following relation hold: $C_3(a(g), u, v) = \det(g)\, C_3(a, x, y) = \det(g)\, C_3(a, g^{-1}(u, v)) = \lambda C_3(a, g^{-1}(u, v))$, where we may consider $\lambda = 1$ (via a time rescaling).

Taking into account that $C_3(x, y) = yp_3(x, y) - xq_3(x, y)$, in [83] were constructed the canonical forms of the cubic homogeneous systems having as the expressions for their polynomials C_3 the indicated canonical forms $(i) - (ix)$:

$$\begin{aligned} x' &= (p+r)x^3 + (s+v)x^2y + qxy^2, \quad C_3 = xy(x-y)(rx+sy),\\ y' &= px^2y + (r+v)xy^2 + (q+s)y^3, \quad rs(r+s) \neq 0; \end{aligned} \tag{2.5}$$

$$\begin{aligned} x' &= (u+1)x^3 + (s+v)x^2y + rxy^2, \qquad C_3 = x(sx+y)(x^2+y^2),\\ y' &= -sx^3 + ux^2y + vxy^2 + (r-1)y^3; \end{aligned} \tag{2.6}$$

$$\begin{aligned} x' &= ux^3 + (p+q+v)x^2y + rxy^2 + qy^3, \quad C_3 = (px^2+qy^2)(x^2+y^2),\\ y' &= -px^3 + ux^2y + vxy^2 + ry^3, \qquad pq > 0; \end{aligned} \tag{2.7}$$

$$\begin{aligned} x' &= 3(u+1)x^3 + (v-1)x^2y + rxy^2, \quad C_3 = x^2y(x-y),\\ y' &= ux^2y + vxy^2 + ry^3; \end{aligned} \tag{2.8}$$

$$\begin{aligned} x' &= ux^3 + (v+1)x^2y + rxy^2, \quad C_3 = x^2(x^2+y^2),\\ y' &= -x^3 + ux^2y + vxy^2 + ry^3; \end{aligned} \tag{2.9}$$

$$\begin{aligned} x' &= (u+1)x^3 + vx^2y + rxy^2, \quad C_3 = x^3y,\\ y' &= ux^2y + vxy^2 + ry^3; \end{aligned} \tag{2.10}$$

$$\begin{aligned} x' &= ux^3 + qx^2y + rxy^2, \quad C_3 = (q-v)x^2y^2,\\ y' &= ux^2y + vxy^2 + ry^3, \quad q - v \neq 0; \end{aligned} \tag{2.11}$$

$$\begin{aligned} x' &= ux^3 + (v+1)x^2y + rxy^2 + y^3, \qquad C_3 = (x^2+y^2)^2,\\ y' &= -x^3 + ux^2y + 3(v-1)xy^2 + ry^3; \end{aligned} \tag{2.12}$$

$$\begin{aligned} x' &= ux^3 + vx^2y + rxy^2, \qquad C_3 = x^4,\\ y' &= -x^3 + ux^2y + vxy^2 + ry^3. \end{aligned} \tag{2.13}$$

Criteria for the existence of an invariant straight line with a given multiplicity. We consider systems (2.1) and their associated vector fields $\mathbb{X} = P(x,y)\dfrac{\partial}{\partial x} + Q(x,y)\dfrac{\partial}{\partial y}$. It is well known that a straight line $L(x,y) = ux + vy + w = 0$, $(u,v) \neq (0,0)$ satisfies $\mathbb{X}(L) = uP(x,y) + vQ(x,y) = (ux + vy + w)R(x,y)$ for some polynomial $R(x,y)$ if and only if it is *invariant* under the flow of the systems. If some of the coefficients u, v, w of an invariant straight line belongs to $\mathbb{C} \setminus \mathbb{R}$, then we say that *the straight line is complex*, otherwise *the straight line is real.*

Definition 2.5 (see [108]). *We say that an invariant affine straight line $L(x,y) = ux+vy+w = 0$ (respectively the line at infinity $Z = 0$) for a cubic vector field $\mathbf{X}$ has **multiplicity m** if there exists a sequence of real cubic vector fields X_k converging to $\mathbf{X}$, such that each X_k has m (respectively $m-1$) distinct invariant affine straight lines $f_k^j = u_k^j x + v_k^j y + w_k^j = 0$, $(u_k^j, v_k^j) \neq (0,0)$, $(u_k^j, v_i^k, w_k^j) \in \mathbb{C}^3$ ($j \in \{1, \ldots m\}$, converging to $f = 0$ as $k \to \infty$ (with the topology of their coefficients), and this does not occur for $m+1$ (respectively m).*

Consider an arbitrary affine line. In a natural way there arises the question: *what are the necessary and sufficient conditions for an arbitrary affine line to be invariant line of the multiplicity k?* In order to construct these conditions we shall use the algebraic method of invariants of differential systems, developed by K.Sibirskii and his disciples (see for instance [122], [136], [98], [6], [37]). We recall further below some results obtained in [108], [109] which will be needed.

Let us consider the polynomials $C_i(a,x,y)$ and $D_i(a,x,y)$, $i = 0,1,2,3$ given in (2.3). We apply a translation $x = x' + x_0$, $y = y' + y_0$ to the polynomials $P(a,x,y)$ and $Q(a,x,y)$ from the right-hand part of (2.1). Therefore we obtain $\tilde{P}(\tilde{a}(a,x_0,y_0),x',y') = P(a,x'+x_0,y'+y_0)$, $\tilde{Q}(\tilde{a}(a,x_0,y_0),x',y') = Q(a,x'+x_0,y'+y_0)$. Let us construct the following polynomials:

$$\Omega_i(a,x_0,y_0) \equiv \mathrm{Res}_{x'}\Big(C_i\big(\tilde{a}(a,x_0,y_0),x',y'\big), C_0\big(\tilde{a}(a,x_0,y_0),x',y'\big)\Big)/(y')^{i+1},$$
$$\tilde{\mathcal{G}}_i(a,x,y) = \Omega_i(a,x_0,y_0)|_{\{x_0=x,\ y_0=y\}} \in \mathbb{R}[a,x,y] \quad (i = 1,2,3)$$

where $\mathrm{Res}_{x'}$ is the resultant of the above polynomials with respect the variable x'.

Notation 2.2. *Assume $\mathcal{G}_i(a,X,Y,Z)$ ($i = 1,2,3$) be the homogenization of $\tilde{\mathcal{G}}_i(a,x,y)$, i.e.*

$$\mathcal{G}_1(a,X,Y,Z) = Z^8\tilde{\mathcal{G}}_1(a,X/Z,Y/Z),\ \mathcal{G}_2(a,X,Y,Z) = Z^{10}\tilde{\mathcal{G}}_2(a,X/Z,Y/Z),$$
$$\mathcal{G}_3(a,X,Y,Z) = Z^{12}\tilde{\mathcal{G}}_3(a,X/Z,Y/Z),$$

and $H(a,X,Y,Z) = \gcd\Big(\mathcal{G}_1(a,X,Y,Z), \mathcal{G}_2(a,X,Y,Z), \mathcal{G}_3(a,X,Y,Z)\Big) \in \mathbb{R}[a,X,Y,Z]$.

The geometrical meaning of the above defined affine comitants is given by the following two lemmas (see [83]):

Lemma 2.3. *The straight line* $L(x,y) \equiv ux + vy + w = 0$, $u, v, w \in \mathbb{C}$, $(u,v) \neq (0,0)$ *is an invariant line for a vector field* $\mathbb{X}$ *if and only if* $L(x,y)$ *is a common factor of the polynomials* $\tilde{\mathcal{G}}_1(a,x,y)$, $\tilde{\mathcal{G}}_2(a,x,y)$ *and* $\tilde{\mathcal{G}}_3(a,x,y)$ *over* $\mathbb{C}$, *i.e.* $\tilde{\mathcal{G}}_i(a,x,y) = (ux+vy+w)\widetilde{W}_i(x,y)$ $(i = 1,2,3)$, *where* $\widetilde{W}_i(x,y) \in \mathbb{C}[x,y]$.

Lemma 2.4. *Consider a cubic system* (2.1) *and let* $a \in \mathbb{R}^{20}$ *be its 20-tuple of coefficients.*

1) If $L(x,y) \equiv ux + vy + w = 0$, $u, v, w \in \mathbb{C}$, $(u,v) \neq (0,0)$ *is an invariant straight line of multiplicity* k *for this system then* $[L(x,y)]^k \mid \gcd(\tilde{\mathcal{G}}_1, \tilde{\mathcal{G}}_2, \tilde{\mathcal{G}}_3)$ *in* $\mathbb{C}[x,y]$, *i.e. there exist* $W_i(a,x,y) \in \mathbb{C}[x,y]$ $(i = 1,2,3)$ *such that*

$$\tilde{\mathcal{G}}_i(a,x,y) = (ux+vy+w)^k W_i(a,x,y), \quad i = 1,2,3. \tag{2.14}$$

2) If the line $l_\infty : Z = 0$ *is of multiplicity* $k > 1$ *then* $Z^{k-1} \mid \gcd(\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3)$, *i.e. we have* $Z^{k-1} \mid H(a,X,Y,Z)$.

We underline that by $P^\star(X,Y,Z), Q^\star(X,Y,Z)$ we denote the homogeneous polynomials associated to the polynomials $P(x,y), Q(x,y)$, i.e.

$$P^\star(X,Y,Z) = Z^n P(X/Z, Y/Z), \quad Q^\star(X,Y,Z) = Z^n Q(X/Z, Y/Z) \tag{2.15}$$

and $C^\star(X,Y,Z) = YP^\star(X,Y,Z) - XQ^\star(X,Y,Z)$.

In order to determine the degree of the common factor of the polynomials $\tilde{\mathcal{G}}_i(a,x,y)$ for $i = 1,2,3$, we shall use the notion of the k^{th} *subresultant* of two polynomials with respect to a given indeterminate (see for instance, [87], [77]).

We say that the k–th *subresultant* with respect to variable z of the two polynomials $f(z)$ and $g(z)$ is the $(m+n-2k) \times (m+n-2k)$ determinant

$$R_z^{(k)}(f,g) = \left| \begin{array}{cccccc} a_0 & a_1 & a_2 & \dots & \dots & a_{m+n-2k-1} \\ 0 & a_0 & a_1 & \dots & \dots & a_{m+n-2k-2} \\ \dots & \dots & \dots & \dots & \dots & \dots\dots\dots \\ 0 & b_0 & b_1 & \dots & \dots & b_{m+n-2k-2} \\ b_0 & b_1 & b_2 & \dots & \dots & b_{m+n-2k-1} \end{array} \right| \begin{array}{l} \left.\begin{array}{l} \\ \\ \end{array}\right\} (m-k) - times \\ \\ \left.\begin{array}{l} \\ \\ \end{array}\right\} (n-k) - times \end{array} \tag{2.16}$$

in which there are $m-k$ rows of a's and $n-k$ rows of b's, and $a_i = 0$ for $i > n$, and $b_j = 0$ for $j > m$.

For $k = 0$ we obtain the standard resultant of two polynomials. In other words we can say that the k–th subresultant with respect to the variable z of the two polynomials $f(z)$ and $g(z)$ can be obtained by deleting the first and the last k rows and columns from its resultant written in the form (2.16) when $k = 0$.

The geometrical meaning of the subresultants is based on the following lemma.

Lemma 2.5. (see [87], [77]). *Polynomials $f(z)$ and $g(z)$ have precisely k roots in common (considering their multiplicities) if and only if the following conditions hold:*

$$R_z^{(0)}(f,g) = R_z^{(1)}(f,g) = R_z^{(2)}(f,g) = \cdots = R_z^{(k-1)}(f,g) = 0 \neq R_z^{(k)}(f,g).$$

For the polynomials in more than one variables it is easy to deduce from Lemma 2.5 the following result.

Lemma 2.6. *Two polynomials $\tilde{f}(x_1, x_2, ..., x_n)$ and $\tilde{g}(x_1, x_2, ..., x_n)$ have a common factor of degree k with respect to the variable x_j if and only if the following conditions are satisfied:*

$$R_{x_j}^{(0)}(\tilde{f},\tilde{g}) = R_{x_j}^{(1)}(\tilde{f},\tilde{g}) = R_{x_j}^{(2)}(\tilde{f},\tilde{g}) = \cdots = R_{x_j}^{(k-1)}(\tilde{f},\tilde{g}) = 0 \neq R_{x_j}^{(k)}(\tilde{f},\tilde{g}),$$

where $R_{x_j}^{(i)}(\tilde{f},\tilde{g}) = 0$ in $\mathbb{R}[x_1, \ldots x_{j-1}, x_{j+1}, \ldots, x_n]$.

2.1.2. The scheme of the proofs of the Main Theorems

Let $L(x, y) = Ux + Vy + W = 0$ be an invariant straight line of cubic systems (S). Then, according to the definition of an invariant line (see p.46), we have $UP(x, y) + VQ(x, y) = (Ux + Vy + W)(Ax^2 + 2Bxy + Cy^2 + Dx + Ey + F)$, and this identity provides the following 10 relations:

$$\begin{aligned}
Eq_1 =& (a_{30} - A)U + b_{30}V = 0,\ Eq_2 = (3a_{21} - 2B)U + (3b_{21} - A)V = 0,\\
Eq_3 =& (3a_{12} - C)U + (3b_{12} - 2B)V = 0,\ Eq_4 = a_{03}U + (b_{03} - C)V = 0,\\
Eq_5 =& (a_{20} - D)U + b_{20}V - AW = 0,\\
Eq_6 =& (2a_{11} - E)U + (2b_{11} - D)V - 2BW = 0,\\
Eq_7 =& a_{02}U + (b_{02} - E)V - CW = 0,\ Eq_8 = (a_{10} - F)U + b_{10}V - DW = 0,\\
Eq_9 =& a_{01}U + (b_{01} - F)V - EW = 0,\ Eq_{10} = a_{00}U + b_{00}V - FW = 0.
\end{aligned} \tag{2.17}$$

Remark 2.1. *Let $C_3 = \prod_{i=1}^{4}(\alpha_i x + \beta_i y), i = 1, 2, 3, 4$. Since infinite singularities of systems* (2.1) *are located on the "ends" of the axes $\alpha_i x + \beta_i y = 0$, the invariant affine lines must be of the form $Ux + Vy + W = 0$, where $U = \alpha_i$ and $V = \beta_i$. In this case, considering W as a parameter, six equations among* (2.17) *become linear with respect to the parameters*

$\{A, B, C, D, E, F\}$ *(with the corresponding non–zero determinant) and we can determine their values, which annulate some of the equations* (2.17)*. So in what follows we will examine only the non-zero equations containing the last parameter* W*.*

According to [4] systems in $\mathbb{CSL}_8$ could not have degenerate infinity because such systems possess at most six invariant lines. So $C_3(x, y) \neq 0$ and hence, at infinity there could be at most four distinct singularities. Therefore considering the fact that in one direction we could have maximum three parallel invariant lines we arrive at following possible configurations (or potential configurations) of invariant straight lines: (3, 3, 1), (3, 2, 2), (3, 2, 1, 1) and (2, 2, 2, 1) (see definitions below).

Consider a system in $\mathbb{CSL}_8$, i.e. this system possesses invariant lines of total multiplicity eight. These lines form a *configuration of type* $(3, 3, 1)$ if there exist two triplets of parallel lines and one additional line, every set with different slopes. And we say that these lines form a *configuration of type* $(3, 2, 1, 1)$ if there exist one triplet and one couple of parallel lines and two additional lines, every set with different slopes. In a similar way are defined *configurations of types* $(3, 2, 2)$ and $(2, 2, 2, 1)$ and these four types of the configurations exhaust all possible configurations formed by 8 invariant lines for a cubic system. Note that in all configurations the invariant straight line which is omitted is the infinite one.

We say that invariant lines of a system in $\mathbb{CSL}_8$ form a *potential configuration of type* $(3, 3, 1)$ (respectively, $(3, 2, 2)$; $(3, 2, 1, 1)$; $(2, 2, 2, 1)$) if there exists a sequence of vector fields $\tilde{D}_k$ as in Definition 2.5 having 7 affine distinct lines of type $(3, 3, 1)$ (respectively, $(3, 2, 2)$; $(3, 2, 1, 1)$; $(2, 2, 2, 1)$).

As it was proved in [27] systems (2.1) with four distinct infinite singularities could not have invariant lines of total multiplicity eight in the configuration of type $(3, 2, 2)$, consequently neither the potential configuration $(3, 2, 2)$. Thus in what follows we will consider only three types of configurations:

$$(i)\ (3,\ 3,\ 1); \quad (ii)\ (3,\ 2,\ 1,\ 1); \quad (iii)\ (2,\ 2,\ 2,\ 1).$$

Following Lemma 2.2 we split the family of cubic systems $\mathbb{CSL}_8$ in 9 subfamilies, according to the number of infinite singularities (real or complex) of systems (2.1) which are determined by the linear factors of the polynomial $C_3(x, y)$. For each one of these subfamilies the proof of the corresponding Main Theorem proceeds in the steps indicated below.

1. First we construct the cubic homogeneous parts $(\tilde{P}_3, \tilde{Q}_3)$ of systems for which the corresponding necessary conditions, provided by Theorem 2.2 of the Thesis are satisfied

in order to have a given number of triplets or/and couples of invariant parallel lines in the respective directions.

2. Secondly, taking cubic systems $\dot{x} = \tilde{P}_3,\ \dot{y} = \tilde{Q}_3$ we we perturb them by adding quadratic, linear and constant terms and using the equations (2.17) we determine these terms in order to get the necessary number of invariant lines in the respective configuration. Thus the second step ends with the construction of the canonical systems possessing the needed configuration. This leads us to the next remark.

   Remark 2.2. *If the perturbed systems have a triplet (respectively a couple) of parallel lines in the direction $Ux + Vy = 0$ then the respective cubic homogeneous systems with right-hand parts $(\tilde{P}_3, \tilde{Q}_3)$ necessarily have the invariant line $Ux + Vy = 0$ of the multiplicity three (respectively two).*

   Thus the second step ends with the construction of the canonical systems possessing the required configuration.

3. The third step consists in the determination of the affine invariant conditions necessary and sufficient for a cubic system to belong to the family of systems (constructed at the second step) which possess the corresponding configuration of invariant lines.

4. And finally, in the case of the existence of multiple invariant lines in a potential configuration we construct the corresponding perturbed systems possessing 8 distinct invariant lines (real and/or complex, including the line at infinity).

2.2. Classification of cubic systems according to their configurations of invariant lines

According to Lemma 2.2 we split the family of cubic systems having 4 distinct infinite singularities into three subfamilies depending on the types of these singularities and namely: systems with four real, systems with two real and two complex and systems with four complex infinite singularities. On the other hand in [27] (p. 1078) we proved that a cubic system (S) with four complex distinct infinite singularities could not have invariant lines of total multiplicity eight. Therefore we examine the first two above mentioned subfamilies and for each one of these subfamilies the proof of the Main Theorem A proceeds in the first 3 steps which were described earlier in Paragraph 2.1.2. We note that the 4th step is trivial in this case because we have only multiple invariant straight lines which could perturb only in parallel lines and the perturbation is very simple to construct. Surely, here we take into account the types of configurations which these systems can possess.

Our main result concerning these two subfamilies of systems is the following one.

Main Theorem A. *Assume that a cubic system possesses invariant lines of total multiplicity 8, including the line at infinity with its own multiplicity. In addition we assume that this system has four distinct infinite singularities. Then:*

I. *The system possesses exactly one of the 17 possible configurations of invariant lines Config. 8.1 – Config. 8.17 given in Figure 2.1.*

II. *This system possesses the specific configuration Config. 8.j* $(j \in \{1, 2, \dots, 17\}$ *if and only if the corresponding conditions included below are fulfilled. Moreover the system can be brought via an affine transformation and time rescaling to the canonical forms, written below next to the configuration:*

1) *Four real distinct infinite singularities* $\Leftrightarrow \mathcal{D}_1 > 0,\ \mathcal{D}_2 > 0,\ \mathcal{D}_3 > 0:$

A_1) *Configuration of type* $(3,3,1)$ $\Leftrightarrow \mathcal{V}_1 = \mathcal{V}_2 = \mathcal{L}_1 = \mathcal{L}_2 = \mathcal{K}_1 = 0,\ \mathcal{K}_2 \neq 0:$

- *Config. 8.1* $\Leftrightarrow$ $\mathcal{K}_3 > 0$*:* $\dot{x} = x(x+1)(x-a),\ \dot{y} = y(y+1)(y-a),\ 0<a\neq 1;$
- *Config. 8.2* $\Leftrightarrow$ $\mathcal{K}_3 < 0$*:* $\dot{x} = x\big[(x+a)^2+1\big],\ \dot{y} = y\big[(y+a)^2+1\big],\ a\neq 0;$
- *Config. 8.3* $\Leftrightarrow$ $\mathcal{K}_3 = 0$*:* $\dot{x} = x^2(1+x),\quad \dot{y} = y^2(1+y).$

A_2) *Configuration of type* $(3,2,1,1)$ $\Leftrightarrow \mathcal{V}_5 = \mathcal{U}_2 = \mathcal{K}_4 = \mathcal{K}_5 = \mathcal{K}_6 = 0,\quad \mathcal{D}_4 \neq 0:$

- *Config. 8.4* $\Leftrightarrow$ $\mathcal{L}_1 \neq 0$ *and* $\mathcal{K}_7 > 0$*:* $\begin{cases} \dot{x} = x(x-1)(x+r), \quad r > 0, \\ \dot{y} = y(y-1)\big[(1-r)x + ry + r\big]; \end{cases}$
- *Config. 8.5* $\Leftrightarrow$ $\mathcal{L}_1 \neq 0$ *and* $\mathcal{K}_7 < 0$*:* $\begin{cases} \dot{x} = x(x-1)(x+r), \quad r < 0, \\ \dot{y} = y(y-1)\big[(1-r)x + ry + r\big]; \end{cases}$
- *Config. 8.6* $\Leftrightarrow$ $\mathcal{L}_1 = 0$*:* $\dot{x} = rx^3,\quad \dot{y} = (r-1)xy^2 + y^3,\ r \neq 0.$

A_3) *Configuration of type* $(2,2,2,1)$ $\Leftrightarrow \mathcal{V}_3 = \mathcal{K}_2 = \mathcal{K}_4 = \mathcal{K}_8 = 0,\quad \mathcal{D}_4 \neq 0:$

- *Config. 8.7* $\Leftrightarrow$ $\mathcal{K}_9 > 0$*:* $\begin{cases} \dot{x} = (x^2-1)(rx + 2y + ry),\ r(r^2-1) \neq 0, \\ \dot{y} = (y^2-1)(x+2rx+y),\ (r+2)(2r+1)\neq 0; \end{cases}$
- *Config. 8.8* $\Leftrightarrow$ $\mathcal{K}_9 < 0$*:* $\begin{cases} \dot{x} = (x^2+1)(rx+2y+ry),\ r(r^2-1) \neq 0, \\ \dot{y} = (y^2+1)(x+2rx+y),\ (r+2)(2r+1)\neq 0; \end{cases}$
- *Config. 8.9* $\Leftrightarrow$ $\mathcal{K}_9 = 0$*:* $\begin{cases} \dot{x} = x^2(rx + 2y + ry),\ r(r^2-1) \neq 0, \\ \dot{y} = y^2(x + 2rx + y),\ (r+2)(2r+1) \neq 0. \end{cases}$

2) *Two real and two complex distinct infinite singularities* $\Leftrightarrow \mathcal{D}_1 < 0$*:*

A_4) *Configuration of type* $(3,3,1)$ $\Leftrightarrow \mathcal{V}_1 = \mathcal{V}_2 = \mathcal{L}_1 = \mathcal{L}_2 = \mathcal{K}_1 = 0,\ \mathcal{K}_2 \neq 0;$

- *Config. 8.10* $\Leftrightarrow$ $\mathcal{K}_3>0$: $\begin{cases}\dot x=(1-r^2)x/4+x^2-y^2+x^3-3xy^2,\\ \dot y=(1-r^2)y/4+2xy+3x^2y-y^3,\ r^2\neq 0, 1/9,\ 1;\end{cases}$

- *Config. 8.11* $\Leftrightarrow$ $\mathcal{K}_3<0$: $\begin{cases}\dot x=(1+r^2)x/4+x^2-y^2+x^3-3xy^2,\\ \dot y=(1+r^2)y/4+2xy+3x^2y-y^3,\quad r\neq 0;\end{cases}$

- *Config. 8.12* $\Leftrightarrow$ $\mathcal{K}_3=0$: $\begin{cases}\dot x=x/4+x^2-y^2+x^3-3xy^2,\\ \dot y=y/4+2xy+3x^2y-y^3.\end{cases}$

A_5) *Configuration of type* $(3,2,1,1)$ $\Leftrightarrow \mathcal{V}_5=\mathcal{U}_2=\mathcal{K}_4=\mathcal{K}_5=\mathcal{K}_6=0,\quad \mathcal{D}_4\neq 0:$

- *Config. 8.13* $\Leftrightarrow$ $\mathcal{L}_1\neq 0$: $\begin{cases}\dot x=(1+r^2)x\big[(x+r)^2+1\big],\quad r\neq 0,\\ \dot y=(1+r^2)^2y+2r(1+r^2)xy-rx^3\\ \qquad +r^2x^2y-rxy^2-y^3;\end{cases}$

- *Config. 8.14* $\Leftrightarrow$ $\mathcal{L}_1=0$: $\begin{cases}x=(1+r^2)x^3,\quad r\neq 0,\\ \dot y=-rx^3+r^2x^2y-rxy^2-y^3.\end{cases}$

A_6) *Configuration of type* $(2,2,2,1)$ $\Leftrightarrow \mathcal{V}_3=\mathcal{K}_2=\mathcal{K}_4=\mathcal{K}_8=0,\quad \mathcal{D}_4\neq 0:$

- *Config. 8.15* $\Leftrightarrow$ $\mathcal{K}_9>0$: $\begin{cases}\dot x=x(x-1)(1+r^2-2x+2ry),\ r\neq 0,\\ \dot y=-(1+r^2)y+(3+r^2)xy-rx^3\\ \qquad -3x^2y-2ry^2+rxy^2-y^3;\end{cases}$

- *Config. 8.16* $\Leftrightarrow$ $\mathcal{K}_9<0$: $\begin{cases}\dot x=2(1+x^2)(ry-x-r),\ r\neq 0,\\ \dot y=r(r^2+3)x+(1-r^2)y-rx^3\\ \qquad -3x^2y+rxy^2-y^3;\end{cases}$

- *Config. 8.17* $\Leftrightarrow$ $\mathcal{K}_9=0$: $\begin{cases}\dot x=-2x^2(x-ry),\ r\neq 0,\\ \dot y=-2ry^2-rx^3-3x^2y+rxy^2-y^3.\end{cases}$

III. *This system could not have a configuration of invariant lines of the type* $(3,3,2)$ *and neither could it have 4 complex* $\mathbb{ISP}$*s.*

Remark 2.3. *If in a configuration an invariant straight line has multiplicity* $k>1$*, then the number* k *appears near the corresponding straight line and this line is in bold face. Real invariant straight lines are represented by continuous lines, whereas complex invariant straight lines are represented by dashed lines. We indicate next to the real singular points of the system, located on the invariant lines, their corresponding multiplicities. In order to describe the various kinds of multiplicity for* $\mathbb{ISP}$*s we use the notation* (a,b)*. By this notation we point out the maximum number* a *(respectively* b*) of infinite (respectively finite) singularities which can be obtained by perturbation of the multiple point.*

The symbols $(\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{K}_8, \mathcal{K}_9)$ used above denote invariant polynomials defined in Paragraph 2.1.1 of the paper.

Corollary. *A cubic system with four distinct infinite singularities possesses the configuration or potential configuration of a given type if and only if the following conditions are satisfied, respectively:*

$$\begin{array}{lcl} (3,3,1) & \Leftrightarrow & \mathcal{V}_1 = \mathcal{V}_2 = \mathcal{L}_1 = \mathcal{L}_2 = \mathcal{K}_1 = 0,\ \mathcal{K}_2 \neq 0; \\ (3,2,1,1) & \Leftrightarrow & \mathcal{V}_5 = \mathcal{U}_2 = \mathcal{K}_4 = \mathcal{K}_5 = \mathcal{K}_6 = 0,\ \ \mathcal{D}_4 \neq 0; \\ (2,2,2,1) & \Leftrightarrow & \mathcal{V}_3 = \mathcal{K}_4 = \mathcal{K}_2 = \mathcal{K}_8 = 0,\ \ \mathcal{D}_4 \neq 0. \end{array}$$

The proof of the Main Theorem A is organized as follows. In Paragraphs 2.2.1 and 2.2.2 we consider two subfamilies of cubic systems with four distinct infinite singularities, and namely, systems with four real and respectively systems with two real and two imaginary singular points at infinity following the fist two steps described in Paragraph 2.1.2. Next, in Subsection 2.3, we construct the necessary and sufficient conditions (mentioned in the Main Theorem A) for the realization of each one of the configurations, constructed in Subsection 2.1.

2.2.1. Cubic systems with four distinct real infinite singularities

Assuming that systems (2.2) possess four distinct real infinite singularities (i.e. the conditions $\mathcal{D}_1 > 0$, $\mathcal{D}_2 > 0$, $\mathcal{D}_3 > 0$ hold), according to Lemma 2.2 via a linear transformations they could be brought to the family of systems

$$\begin{aligned} \dot{x} &= a + cx + dy + gx^2 + 2hxy + ky^2 + \tilde{P}_3(x,y), \\ \dot{y} &= b + ex + fy + lx^2 + 2mxy + ny^2 + \tilde{Q}_3(x,y) \end{aligned} \tag{2.18}$$

where $\tilde{P}_3(x,y) = (p+r)x^3 + (s+v)x^2y + qxy^2$, $\quad \tilde{Q}_3(x,y) = px^2y + (r+v)xy^2 + (q+s)y^3$ for which we have $C_3 = xy(x-y)(rx+sy)$ and

$$rs(r+s) \neq 0. \tag{2.19}$$

Systems with configuration (3, 3, 1). Since we have two triplets of parallel invariant lines, according to Theorem 2.2 the conditions $\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{U}_1 = 0$ are necessary for systems (2.18). Moreover in [83, Section 5.1] it was proved that in this case via a linear transformation and time rescaling the cubic homogeneities of these systems could be brought to the forms:

$$\tilde{P}_3 = x^3, \quad \tilde{Q}_3 = y^3. \tag{2.20}$$

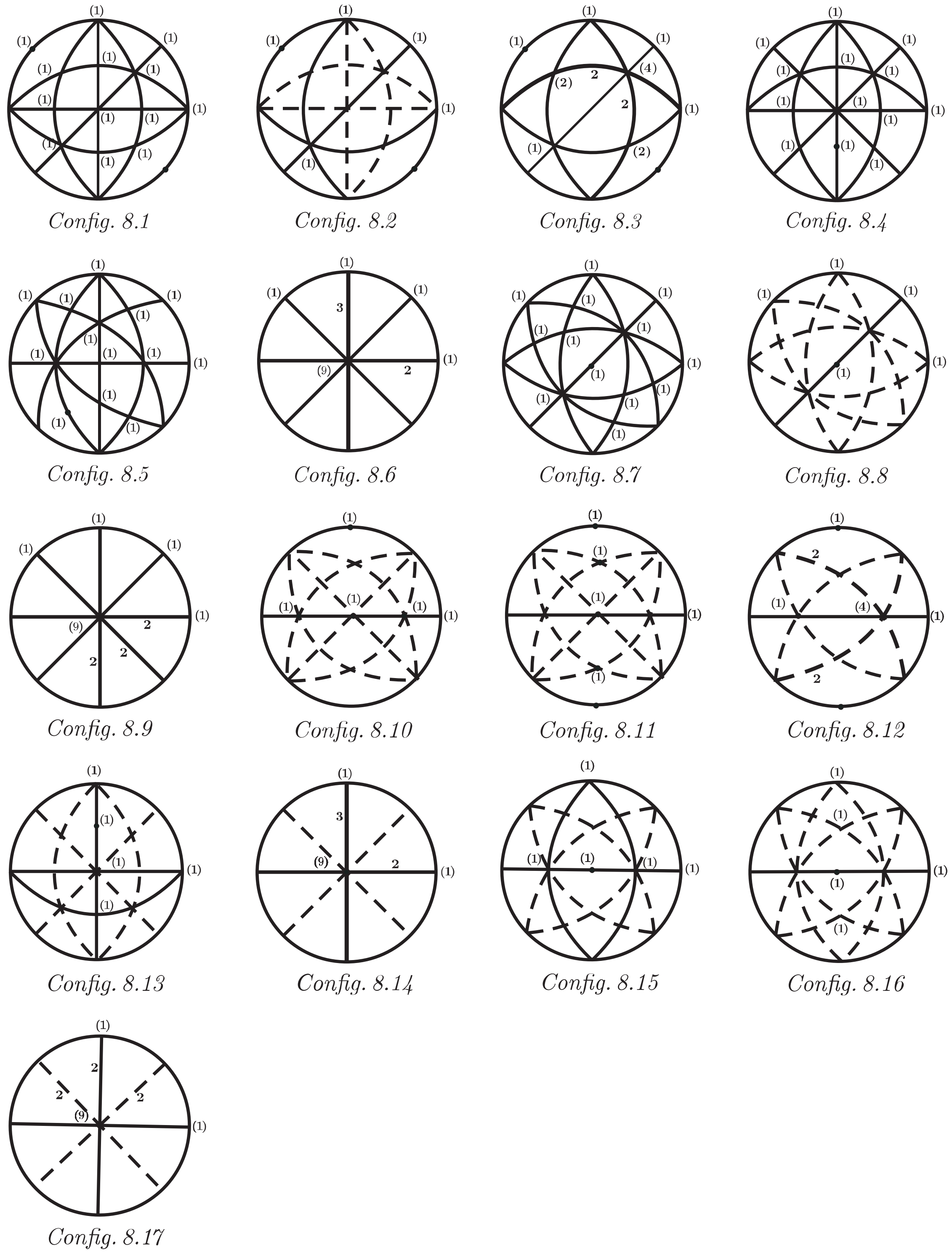

Fig. 2.1. Configurations of invariant lines for systems in $\mathbb{CSL}_8$ with 4 $\mathbb{ISP}s$

So applying a translation we may assume $g = n = 0$ in the quadratic parts of systems (2.18) with the cubic homogeneities of the form (2.20). In such a way we get the family of systems

$$\dot{x} = a + cx + dy + 2hxy + ky^2 + x^3, \quad \dot{y} = b + ex + fy + lx^2 + 2mxy + y^3 \qquad (2.21)$$

for which we calculate $C_3(x, y) = xy(x - y)(x + y)$.

In order to find out the directions of two triplets according to Remark 2.2 we determine the multiplicity of the invariant lines of system (2.20). For this system we calculate (see the definition of the polynomial $H(X,Y,Z)$ on the page 47, Notation 2.2): $H(X,Y,Z) = \gcd(\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_3) = 3X^3(X-Y)Y^3(X+Y)$. So (2.20) possesses two triple invariant lines $x=0$ and $y=0$ and by Remark 2.2 systems (2.21) could have triplets of parallel invariant lines only in these two directions.

(i) The direction $x=0$. Considering equations (2.17) and Remark 2.1 we obtain

$$Eq_7 = k,\ Eq_9 = d - 2hW,\ Eq_{10} = a - cW - W^3 \tag{2.22}$$

and obviously we can have a triplet of parallel invariant lines (which could coincide) in the direction $x=0$ if and only if $k=d=h=0$. Assuming that these conditions hold we consider the another direction for the second triplet.

(ii) The direction $y=0$. In this case we have

$$Eq_5 = l,\ Eq_8 = e - 2mW,\ Eq_{10} = b - fW - W^3 \tag{2.23}$$

and again we conclude that for the existence of three parallel invariant lines in the direction $y=0$ for systems (2.21) the conditions $e=l=m=0$ have to be satisfied.

It remains to examine the directions $y=x$ and $y=-x$ in order to determine the conditions for the existence of exactly one invariant line in one of these two directions.

For the *direction* $y=x$ we have

$$Eq_7 = -3W,\ Eq_9 = f - c - 3W^2,\ Eq_{10} = a + b - cW - W^3$$

whereas for the *direction* $y=-x$ we obtain

$$Eq_7 = -3W,\ Eq_9 = c - f + 3W^2,\ Eq_{10} = a - b - cW - W^3.$$

We observe that in each one of the cases we could have only one invariant line. Moreover the necessary and sufficient conditions for the existence of such a line are $c-f=a+b=0$ in the first case and $c-f=a-b=0$ in the second case.

Thus we conclude that for the existence of a single invariant line in one of the mentioned directions the following conditions are necessary and sufficient:

$$c - f = (a-b)(a+b) = 0, \quad a^2 + b^2 \neq 0.$$

Since the respective family of systems is of the form

$$\dot{x} = a + cx + x^3, \quad \dot{y} = b + cy + y^3 \tag{2.24}$$

we may assume $b = a \neq 0$ due to the rescaling $y \to -y$ in the case $b = -a$ and we arrive at the family of systems

$$\dot{x} = a + cx + x^3, \quad \dot{y} = a + cy + y^3. \tag{2.25}$$

These systems possess the invariant lines defined by the equations

$$x^3 + cx + a = 0, \quad x - y = 0, \quad y^3 + cy + a = 0.$$

Since the discriminant of the polynomial $x^3 + cx + c$ equals $\xi = -(27a^2 + 4c^3)$ (and clearly it coincides with the discriminant of the polynomial $y^3 + cy + a$) we conclude that the systems above possess 7 invariant affine lines which are as follows:

$\xi > 0 \quad \Rightarrow \quad$ 7 real simple distinct;

$\xi < 0 \quad \Rightarrow \quad$ 3 real and 4 complex all simple distinct;

$\xi = 0 \quad \Rightarrow \quad$ 3 simple and 2 double all real distinct.

As we have two triplets of parallel invariant lines it is clear that all 9 finite singularities (real and/or complex) are located at the intersections of these lines. It remains to observe that in the case of 4 complex lines we have only one real finite singularity: the intersection of the three real lines.

Thus we obtain the configuration given by *Config. 8.1* if $\xi > 0$, by *Config. 8.2* if $\xi < 0$ and by *Config. 8.3* if $\xi = 0$ (see Figure 2.1).

Systems with configuration (3, 2, 1, 1). First we need to construct the cubic homogeneous parts $(\tilde{P}_3, \tilde{Q}_3)$ of systems (2.18) for which the conditions $\mathcal{V}_4 = \mathcal{V}_5 = \mathcal{U}_2 = 0$ are fulfilled. According to [27, Section 3.3.1] the condition $\mathcal{V}_5 = \mathcal{U}_2 = 0$ implies $\mathcal{V}_4 = 0$. More exactly, we have the next remark.

Remark 2.4. *In order to construct the whole class of systems possessing the configuration or potential configuration* $(3, 2, 1, 1)$ *it is sufficient to consider the family of cubic systems with the homogeneous cubic parts of the forms*

$$\dot{x} = rx^3, \qquad \dot{y} = (r-1)xy^2 + y^3. \tag{2.26}$$

Since $r \neq 0$, due to a translation we may assume $g = n = 0$ in the quadratic parts of systems (2.18) with the cubic homogeneities of the form (2.26). Considering Remark 2.4 we get the next result.

Lemma 2.7. *Assume that a cubic system* (2.18) *possesses* 7 *invariant affine straight lines with configuration or potential configuration* (3, 2, 1, 1)*. Then via an affine transformation and a time rescaling this system could be brought to a system belonging to the following family:*

$$\begin{aligned}\dot{x} =& a + cx + dy + 2hxy + ky^2 + rx^3, \quad r(r+1) \neq 0,\\ \dot{y} =& b + ex + fy + lx^2 + 2mxy + (r-1)xy^2 + y^3.\end{aligned} \tag{2.27}$$

In what follows we shall determine necessary and sufficient conditions for a system (2.27) to have a configuration or potential configuration (3, 2, 1, 1). Considering Remark 2.2 for the homogeneous systems (2.26) corresponding to (2.27) we calculate

$$H(X, Y, Z) = \gcd(\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3) = X^3(X - Y)Y^2(rX + Y). \tag{2.28}$$

So the invariant line $x = 0$ (respectively $y = 0$) of systems (2.26) is of multiplicity three (respectively two). Hence by Remark 2.2 the systems (2.27) could possess one triplet (respectively one couple) of invariant lines in the direction $x = 0$ (respectively $y = 0$). However for some values of the parameter r the common divisor $\gcd(\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3)$ could contain additional factors. To detect them we calculate: $Res_X(\mathcal{G}_2/H,\, \mathcal{G}_1/H) = (r-1)(2+r)(1+2r)Y^3$, $Res_X(\mathcal{G}_3/H,\, \mathcal{G}_1/H) = (r-1)(2+r)(1+2r)(4-r)(4r-1)Y^5$, $Res_Y(\mathcal{G}_2/H,\, \mathcal{G}_1/H) = (r-1)(2+r)(1+2r)X^3$, $Res_Y(\mathcal{G}_3/H,\, \mathcal{G}_1/H) = (r-1)(2+r)(1+2r)(r-4)(4r-1)X^3$. Therefore in order to have a nonconstant common factor of the polynomials $\mathcal{G}_1/H$, $\mathcal{G}_2/H$ and $\mathcal{G}_3/H$ the condition $(r-1)(2+r)(1+2r) = 0$ has to be satisfied. However in [27] (see pages 1052-1053) it was proved that in this case systems (2.27) could not have 7 affine invariant straight lines.

So we assume $(r-1)(1+2r)(2+r) \neq 0$ and we shall examine all four directions $(x = 0, y = 0, y = x, y = -rx)$ defined by the factors of $C_3(x, y)$.

(i) The direction $x = 0$*.* Considering the equations (2.17) and Remark 2.1 we obtain

$$Eq_7 = k,\ Eq_9 = d - 2hW,\ Eq_{10} = a - cW - rW^3 \tag{2.29}$$

and obviously we can have a triplet of parallel invariant line (which could coincide) in the direction $x = 0$ if and only if $k = d = h = 0$.

(ii) The direction $y = 0$*.* In this case considering the equations (2.17) and above conditions we have

$$Eq_5 = l, Eq_8 = e - 2mW + (r-1)W^2,\ Eq_{10} = b - fW - W^3. \tag{2.30}$$

So we conclude that for the existence of a couple of parallel invariant lines for systems (2.27) in this direction it is necessary and sufficient $l = 0$ and $R^{(0)}_W(Eq_8, Eq_{10}) = R^{(1)}_W(Eq_8, Eq_{10}) = 0$.

We calculate $R_W^{(1)}(Eq_8, Eq_{10}) = -4m^2 + e(r-1) - f(r-1)^2 = 0$ and as $r \neq 1$ we have $e = \big[4m^2 + f(r-1)^2\big]/(r-1)$. Then we obtain $R_W^{(0)}(Eq_8, Eq_{10}) = [8m^3 + 2fm(r-1)^2 + b(r-1)^3]/(r-1)^3 = 0$ and hence we get $b = -2m\big[4m^2 + f(r-1)^2\big]/(r-1)^3$.

Thus in order to have a triplet in the direction $x = 0$ and a couple in the direction $y = 0$ in the case $r-1 \neq 0$ the following conditions are necessary and sufficient for systems (2.27):

$$k=d=h=l=0,\;\; e=\big[4m^2+f(r-1)^2\big]/(r-1),\;\; b=-2m\big[4m^2+f(r-1)^2\big]/(r-1)^3. \tag{2.31}$$

(iii) The direction $y = x$. In this case we have

$$\begin{gathered} Eq_7=l-2h-k+2m-(1+2r)W=0,\;\; Eq_9=e+f-c-d+2(l-h+m)W-3rW^2=0,\\ Eq_{10}=b-a+(e-c)W+lW^2-rW^3=0. \end{gathered} \tag{2.32}$$

Since $2r+1 \neq 0$ these equations could have only one common solution. Considering the conditions (2.31) the equation $Eq_7 = 0$ gives $W = 2m/(2r+1)$. Then the equations $Eq_9 = 0$ and $Eq_{10} = 0$ yield

$$c = fr + \frac{12m^2r(2+r)}{(-1+r)(1+2r)^2},\quad a = -\frac{6fmr}{(r-1)(1+2r)} - \frac{72m^3r(1+r+r^2)}{(r-1)^3(1+2r)^3}. \tag{2.33}$$

(iv) The direction $rx + y = 0$. In this case we have

$$\begin{gathered} Eq_5=l-2mr-2hr^2+kr^3-r(2+r)W,\;\; Eq_{10} = b+ar-(f+dr)W+krW^2-W^3,\\ Eq_8=e+cr-fr-dr^2-2(m+hr-kr^2)W-(1+2r)W^2. \end{gathered} \tag{2.34}$$

Since $r(r+2) \neq 0$, considering the conditions (2.31) and (2.33) the equation $Eq_5 = 0$ gives $W = -2m/(r+2)$. Then the equations $Eq_8 = 0$ and $Eq_{10} = 0$ take the form

$$Eq_8 = (r+1)U(f,m,r) = 0,\quad Eq_{10} = -\frac{6m(1+r)(1+r+r^2)U(f,m,r)}{(r-1)(2+r)(1+2r)} = 0$$

where $U(f,m,r) = f(r-1) + [12m^2(1+5r+15r^2+5r^3+r^4)]/[(r-1)(2+r)^2(1+2r)^2]$. Since $r+1 \neq 0$ the condition $Eq_8 = 0$ gives $U(f,m,r) = 0$ and then $Eq_{10} = 0$. In this case the condition $U(f,m,r) = 0$ implies $f = \dfrac{12m^2(1+5r+15r^2+5r^3+r^4)}{(r-1)^2(2+r)^2(1+2r)^2}$ and taking into account (2.31) and (2.33) we arrive at the following relations among the parameters of systems (2.27) in the case $(r-1)(2+r)(1+2r) \neq 0$:

$$\begin{gathered} k=d=h=l=0,\quad f=-\frac{12m^2(1+5r+15r^2+5r^3+r^4)}{(r-1)^2(2+r)^2(1+2r)^2},\\ b=-\frac{8m^3(1+7r+r^2)}{(r-1)(2+r)^2(1+2r)^2},\quad c=-\frac{108m^2r(1+r+r^2)}{(r-1)^2(2+r)^2(1+2r)^2},\\ a=\frac{216m^3r}{(r-1)^2(2+r)^2(1+2r)^2},\quad e=\frac{4m^2(r-1)(1+7r+r^2)}{(2+r)^2(1+2r)^2}. \end{gathered} \tag{2.35}$$

So we obtain the following 2-parameter family of systems

$$\dot{x} = r\Big[x + \frac{6m}{(r-1)(2+r)}\Big]\Big[x - \frac{6m}{(r-1)(1+2r)}\Big]\Big[x - \frac{6m}{(1+2r)(2+r)}\Big],$$
$$\dot{y} = \Big[y + \frac{2m(r-1)}{(2+r)(1+2r)}\Big]\Big[y + \frac{2m(1+7r+r^2)}{(r-1)(2+r)(1+2r)}\Big]\Big[y + (r-1)x - \frac{2m}{(r-1)}\Big].$$

Since $(r-1)(2+r)(1+2r) \neq 0$ we set a new parameter u as follows: $m = \frac{u}{6}(r-1)(2+r)(1+2r)$ and this leads to the following systems:

$$\begin{aligned} &\dot{x} = r\big[x + u(1+2r)\big]\big[x - u(2+r)\big]\big[x - u(r-1)\big],\\ &\dot{y} = \big[y + u(r-1)^2/3\big]\big[y + u(1+7r+r^2)/3\big]\big[y + (r-1)x - u(2+r)(1+2r)/3\big]. \end{aligned} \tag{2.36}$$

Assume first that $u \neq 0$. Since $r \neq 0$ by means of the transformation $x_1 = \frac{x}{3u} + \frac{1-r}{3}$, $y_1 = -\frac{y}{3ru} - \frac{(r-1)^2}{9r}$, $t_1 = 9ru^2 t$ systems (2.36) become the systems (we keep the old notations for variables)

$$\dot{x} = x(x-1)(x+r), \qquad \dot{y} = y(y-1)\big[(1-r)x + ry + r\big]. \tag{2.37}$$

We observe that the above systems possess seven invariant affine lines $L_1 = x$, $L_2 = x-1$, $L_3 = x+r$, $L_4 = y$, $L_5 = y-1$, $L_6 = x-y$, $L_7 = x+ry$ in the configuration $(3,2,1,1)$. Since $r(r+1) \neq 0$ we conclude that we could not have coinciding invariant lines.

On the other hand systems (2.37) possess 9 finite singularities: $(0,0), (0,1), (0,-1), (1,0), (-r,0)$, $(1,1), (1,-1/r), (-r,1), (-r,-r)$. We observe that 8 singular point are located at the intersections of the invariant lines, whereas the ninth one (and namely, (0,-1)) is located on the invariant line $L_1 = 0$. Moreover, the positions of the invariant lines which form the triplet depends on the parameter r. More precisely, if $r > 0$ then the line L_1 is placed between the parallel invariant lines L_2 and L_3 and in the case $r < 0$ the lines L_2 and L_3 are located on the right with respect to L_1. Thus taking into consideration that on the line L_1 it is located the unique point of the intersection of 4 invariant lines L_1, L_4, L_6 and L_7 (the origin of coordinate), we arrive at two different configurations. Namely we obtain the configuration given by *Config. 8.4* if $r > 0$ and by *Config. 8.5* if $r < 0$ (see Figure 2.1).

Assume now $u = 0$. Then systems (2.36) become the homogeneous systems (2.26) possessing invariant lines $x = 0$ (triple), $y = 0$ (double), $y = x$ and $y = -rx$ (see (2.28)). This leads to the configuration *Config. 8.6* (see Figure 2.1).

Thus we arrive at the next result.

Lemma 2.8. *A system* (2.27) *possesses the configuration or potential configuration of invariant lines of the type* $(3,2,1,1)$ *only in the case* $(r-1)(1+2r)(2+r) \neq 0$. *Moreover if* *this system possesses such a configuration then it could be written either in the form* (2.37) *or* (2.26)*.*

Systems with configuration (2, 2, 2, 1). As a first step we need to construct the cubic homogeneous parts $(\tilde{P}_3, \tilde{Q}_3)$ of systems (2.18) for which we force the condition $\mathcal{V}_3 = 0$. In [27, Section 3.4.1] it was proved that in this case the cubic homogeneities could be brought to the form:

$$\dot{x} = rx^3 + (2+r)x^2y, \quad \dot{y} = (1+2r)xy^2 + y^3. \tag{2.38}$$

Since in systems (2.18) with the homogenous cubic parts of the form (2.38) due to a translation we may assume $n = 0$, we arrive at the next result.

Lemma 2.9. *Assume that a cubic system* (2.18) *possesses* 7 *invariant affine straight lines with configuration or potential configuration* $(2, 2, 2, 1)$*. Then via an affine transformation and a time rescaling this system could be brought to a system belonging to the following family:*

$$\begin{aligned} \dot{x} =& a + cx + dy + gx^2 + 2hxy + ky^2 + rx^3 + (2+r)x^2y, \\ \dot{y} =& b + ex + fy + lx^2 + 2mxy + (1+2r)xy^2 + y^3, \quad r(r+1) \neq 0. \end{aligned} \tag{2.39}$$

In what follows we shall determine necessary and sufficient conditions for a system (2.39) to have a configuration or potential configuration $(2, 2, 2, 1)$. Considering Remark 2.2 for the homogeneous systems (2.38), corresponding to systems (2.39) we calculate

$$H(X, Y, Z) = \gcd(\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3) = X^2(X-Y)Y^2(rX+Y)^2. \tag{2.40}$$

So each one of the invariant lines $x = 0$, $y = 0$ and $rx + y = 0$ of systems (2.38) is of multiplicity two and in the direction $y = x$ there exists one line.

We claim that in order to have exactly three couples of invariant straight lines, for systems (2.39) the condition $(r+2)(2r+1)(r-1) \neq 0$ must hold. Indeed using the equations (2.17) we evaluate them for each one of the four directions.

(i) For the direction $x = 0$ we obtain

$$Eq_7 = k, \; Eq_9 = d - 2hW + (2+r)W^2, \; Eq_{10} = a - cW + gW^2 - rW^3 \tag{2.41}$$

and to have exactly two parallel invariant lines in this direction the condition $r + 2 \neq 0$ is necessary.

(ii) For the direction $y = 0$ we have

$$Eq_5 = l, \; Eq_8 = e - 2mW + (1+2r)W^2, \; Eq_{10} = b - fW - W^3. \tag{2.42}$$

and it is evident, the condition $1+2r\neq 0$ must be satisfied (in order to have exactly two parallel invariant lines in this direction).

(iii) For the direction $y=x$ we calculate

$$\begin{gathered} Eq_6 = l-g-2h-k+2m-3(1+r)W,\ Eq_8 = ef-c-d+(l-g+k)W+(1-r)W^2,\\ Eq_{10} = -a+b+dW-fW-kW^2-W^3. \end{gathered} \tag{2.43}$$

and as $r+1\neq 0$ in this direction could be at most one invariant line.

(iv) For the direction $y=-rx$ we obtain

$$\begin{gathered} Eq_5 = l+(g-2m)r-2hr^2+kr^3,\quad Eq_{10} = b+ar-(f+dr)W+krW^2-W^3,\\ Eq_8 = e+(c-f)r-dr^2-2(m+2hr-2kr^2)W+(1-r)W^2. \end{gathered} \tag{2.44}$$

We observe, that to have exactly two invariant lines in this direction it is necessary $r-1\neq 0$.

So we conclude that the three needed couples of parallel invariant lines could be only in the directions $x=0$, $y=0$ and $y=-rx$ and for this the condition $(r+2)(2r+1)(r-1)\neq 0$ must hold. So our claim is proved.

Since $r+2\neq 0$ without loss of generality in systems (2.39) we may assume $h=0$ due to the translation $x=x_1+h(1+2r)/(3(2+r))$, $y=y_1-h/(2+r)$, which conserves the previous relation $n=0$. So we have to force the existence of parallel lines in the above mentioned directions. Considering (2.41), (2.42), (2.44) and $h=0$ we obtain: $k=l=0$, $r(g-2m)=0$ and this implies $g=2m$.

Now we look for the sufficient conditions under the parameters of systems (2.39) for the existence of three couples of parallel lines, assuming that the following conditions hold:

$$k=l=h=0,\quad g=2m. \tag{2.45}$$

(i) Direction $x=0$. Considering (2.41) we get $Eq_9=d+(2+r)W^2=0$, $Eq_{10}=a-cW+2mW^2-rW^3=0$ and by Lemma 2.5 in order to have two common solutions the following conditions are necessary and sufficient: $R_W^{(0)}(Eq_9,Eq_{10})=R_W^{(1)}(Eq_9,Eq_{10})=0$. Since $R_W^{(1)}(Eq_9,Eq_{10})=-(2+r)(2c+cr-dr)$ and $r(r+2)\neq 0$ we obtain $d=c(2+r)/r$ and we calculate $R_W^{(0)}(Eq_9,Eq_{10})=(2+r)^3(2cm-ar)^2/r^2=0$. Therefore we get $a=2cm/r$ and this implies $Eq_9=(2+r)(c+rW^2)/r$, $Eq_{10}=(2m-rW)(c+rW^2)/r$ and hence, we have two common solutions, which could be real or complex, distinct or coinciding. On the other hand for the parameters of systems (2.39) we obtain the following relations:

$$(r-1)(r+2)(2r+1)\neq 0,\ k=l=h=0,\ g=2m,\quad d=c(2+r)/r,\ a=2cm/r. \tag{2.46}$$

(ii) Direction $y = 0$. Taking into account (2.42) and (2.46) in this case we obtain $Eq_8 = e - 2mW + (1 + 2r)W^2 = 0$, $Eq_{10} = b - fW - W^3 = 0$ and $R_W^{(1)}(Eq_8 Eq_{10}) = e(1 + 2r) - f(1 + 2r)^2 - 4m^2 = 0$ and this implies $e = f(1 + 2r) + 4m^2/(1 + 2r)$. Then we calculate $R_W^{(0)}(Eq_8, Eq_{10}) = \big[8m^3 + 2fm(1+2r)^2 + b(1+2r)^3\big]^2/(1+2r)^3 = 0$ and hence we get $b = -2\big[4m^3 + fm(1 + 2r)^2\big]/(1 + 2r)^3$. Considering the new obtained conditions we arrive at the following relations among the parameters of systems (2.39):

$$\begin{gathered}(r-1)(r+2)(2r+1) \neq 0, \quad k = l = h = 0, \quad g = 2m, \; d = c(2+r)/r, \quad a = 2cm/r, \\ e = f(1+2r) + 4m^2/(1+2r), \; b = -2\big[4m^3 + fm(1+2r)^2\big]/(1+2r)^3\end{gathered} \tag{2.47}$$

(iii) Direction $y = -rx$. Considering (2.44) we get

$$\begin{aligned}Eq_8 &= (1 + r)(f - cr) + 4m^2/(1 + 2r) - 2mW - (r - 1)W^2 = 0,\\ Eq_{10} &= 2cm - 8m^3/(1 + 2r)^3 - 2fm/(1 + 2r) - (2c + f + cr)W - W^3 = 0,\\ R_W^{(1)}(Eq_8, Eq_{10}) &= 2c(r - 1) - 2fr(r - 1) - 12m^2r/(1 + 2r) = 0\end{aligned}$$

and therefore we obtain $c = fr + 6m^2r/[(r - 1)(1 + 2r)]$. Then we calculate

$$R_W^{(0)}(Eq_8, Eq_{10}) = -\frac{144m^2r^2(1+r)^2\big[f(r-1)^2(1+2r)^2 + 3m^2(1-2r+4r^2)\big]^2}{(2+r)^6(1+2r)^4} = 0$$

and clearly we have either $m = 0$ or $m \neq 0$ and $f = -3m^2(1-2r+4r^2)/[(r-1)^2(1+2r)^2]$.

1) *The case $m = 0$.* Then by (2.47) we get the conditions

$$\begin{gathered}(r-1)(r+2)(2r+1) \neq 0, \quad k = l = h = g = m = a = b = 0,\\ d = f(2+r), \quad e = f(1+2r), \quad c = fr\end{gathered} \tag{2.48}$$

and for the direction $y = -rx$ we obtain $Eq_8 = (1 - r)(f + 2fr + fr^2 + W^2)$, $Eq_{10} = -W(f + 2fr + fr^2 + W^2)$. So we have two common solutions, which could be real or complex, distinct or coinciding. In this case for the last direction (i.e. $y = x$) we calculate $Eq_6 = -3(1 + r)W$, $Eq_8 = (1 - r)W^2$, $Eq_{10} = W(f + fr - W^2)$ and the common solution is $W = 0$. Thus we get the family of systems

$$\dot{x} = (f + x^2)(rx + 2y + ry), \quad \dot{y} = (f + y^2)(x + 2rx + y) \tag{2.49}$$

with the condition $r(r^2 - 1)(r + 2)(2r + 1) \neq 0$ and $f \in \{-1, 0, 1\}$ due to the rescaling $(x, y, t) \mapsto (|f|^{1/2}x,\ |f|^{1/2}y,\ t/|f|)$ if $f \neq 0$. These systems possess the following invariant lines: $x^2 + f = 0$, $y^2 + f = 0$, $y - x = 0$, $(rx + y)^2 + f(1 + r)^2 = 0$. We observe that the line $y = x$ is real and all other lines are distinct real (respectively complex) if $f < 0$ (respectively $f > 0$) and we have three double invariant lines in the case $f = 0$.

Thus we obtain the configurations either *Config. 8.7* if $f < 0$ or *Config. 8.8* if $f > 0$ or *Config. 8.9* if $f = 0$ (see Figure 2.1).

2) *The case* $f = -\dfrac{3m^2(1-2r+4r^2)}{(-1+r)^2(1+2r)^2}$ *and* $m \neq 0$. We claim that in this case in the direction $y = x$ we could not have any invariant line. Indeed, considering (2.43) we obtain

$$Eq_6 = -3(1+r)W = 0, \quad Eq_8 = -8m^2(2+r)/[(r-1)(1+2r)] - 2mW + (1-r)W^2 = 0$$

and we observe that due to $r + 1 \neq 0$ these equations could have only the common solution $W = 0$. However in this case we must have $m(2 + r) = 0$ which contradicts $m(2 + r) \neq 0$. So our claim is proved.

2.2.2. Cubic systems with 2 real and 2 complex infinite singularities

According to Lemma 2.2 in this case the condition $\mathcal{D}_1 < 0$ holds and the systems (2.2) due to a linear transformation and time rescaling could be brought to the systems

$$\begin{aligned} \dot{x} &= a + cx + dy + gx^2 + 2hxy + ky^2 + (u+1)x^3 + (s+v)x^2y + rxy^2, \\ \dot{y} &= b + ex + fy + lx^2 + 2mxy + ny^2 - sx^3 + ux^2y + vxy^2 + (r-1)y^3. \end{aligned} \tag{2.50}$$

For these systems we have $C_3 = x(sx + y)(x^2 + y^2)$ and hence, infinite singular points are situated at the "ends" of the straight lines: $x = 0$, $y = -sx$ and $y = \pm ix$. In what follows we split our examination in three cases depending on the type of configuration of invariant straight lines which these systems can possess.

Systems with configuration (3, 3, 1). Since we have two triplets of parallel invariant lines, according to Theorem 2.2 the conditions $\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{U}_1 = 0$ are necessary for systems (2.50). Moreover in [83, Section 6.1] it was proved that providing the conditions above, a cubic homogenous system with two real and two complex (all distinct) infinite singularities via a linear transformation and time rescaling could be brought either to the system

$$\dot{x} = x^3, \quad \dot{y} = -y^3 \ \text{ if } \mathcal{L}_4 < 0,$$

or to the system

$$\dot{x} = x^3 - 3xy^2, \quad \dot{y} = 3x^2y - y^3 \ \text{ if } \mathcal{L}_4 > 0.$$

For the first system we calculate $H(\tilde{a}, X, Y, Z) = 3X^3Y^3(X^2+Y^2)$, $C_3(x, y) = xy(x^2+y^2)$, whereas for the second one we have $H(\tilde{a}, X, Y, Z) = 6XY(X^2+Y^2)^3$, $C_3(x, y) = -2xy(x^2+y^2)$. In the first case we must have two triplets of parallel lines in the real directions and hence, forcing the existence of a line in the complex direction we get 8 invariant affine lines.

Thus it remains to consider only the systems with cubic homogeneities of the second type. We observe that due to a translation we may assume $g = n = 0$ in the quadratic parts of the systems (2.50) and so we examine the family of systems

$$\begin{aligned} \dot{x} &= a + cx + dy + 2hxy + ky^2 + x^3 - 3xy^2, \\ \dot{y} &= b + ex + fy + lx^2 + 2mxy + 3x^2y - y^3. \end{aligned} \tag{2.51}$$

(i) The direction $x + iy = 0$. In this case we obtain

$$\begin{aligned} Eq_{10} = a + ib - (c + ie)W + ilW^3 - W^3,\ Eq_6 = l + 2h + i(k + 2m), \\ Eq_9 = d + e + i(f - c) - (l - ik)W. \end{aligned} \tag{2.52}$$

As all the parameters of systems (2.51) are real we conclude, that to have exactly three parallel invariant lines (which could coincide) in this direction it is necessary and sufficient to be satisfied the conditions $l = k = h = m = 0$, $d = -e$ and $f = c$.

Thus we arrive to the family of systems

$$\dot{x} = a + cx - ey + x^3 - 3xy^2, \quad \dot{y} = b + ex + cy + 3x^2y - y^3 \tag{2.53}$$

for which we shall examine simultaneously the real directions: $x = 0$ and $y = 0$.

For *the direction $x = 0$* we calculate

$$Eq_7 = 3W, \quad Eq_9 = -e, \quad Eq_{10} = a - cW - W^3, \tag{2.54}$$

whereas for *the direction $y = 0$* we have

$$Eq_5 = -3W, \quad Eq_8 = e, \quad Eq_{10} = b - cW + W^3. \tag{2.55}$$

We observe that in each one of the cases we could have only one invariant line defined by $W = 0$. Moreover the necessary and sufficient conditions for the existence of such line are $e = a = 0$ in the first case and $e = b = 0$ in the second case. We conclude that for the existence of exactly one invariant line in one of the real directions for systems (2.53), the following conditions are necessary and sufficient: $e = ab = 0, \quad a^2 + b^2 \neq 0$. Consequently we arrive at systems for which we may assume $b = 0$ due to the change $(x, y, t) \mapsto (y, x, -t)$ in the case $a = 0$. So we get the family of systems

$$\dot{x} = a + cx + x^3 - 3xy^2, \quad \dot{y} = cy + 3x^2y - y^3 \tag{2.56}$$

possessing the following invariant straight lines:

$$y = 0, \quad (x + iy)^3 + c(x + iy) + a = 0, \quad (x - iy)^3 + c(x - iy) + a = 0.$$

Since the discriminant of the cubic polynomial $\phi(z) = z^3 + cz + a$, where $z = x \pm iy$ equals $\xi = -(27a^2 + 4c^3)$, we conclude that the above systems possess 7 invariant affine lines (considered with their multiplicities), which are as follows:

$\xi \neq 0 \quad \Rightarrow \quad$ one real simple and 6 complex distinct simple;

$\xi = 0 \quad \Rightarrow \quad$ one real simple, two complex simple and 2 complex double, all distinct.

As we have two triplets of parallel complex invariant lines it is clear that all 9 finite singularities (real and/or complex) are located at the intersections of these lines. Moreover, as there exist three pair of complex conjugate lines we have three real finite singularities, which are distinct if $\xi \neq 0$ and two of them coincide if $\xi = 0$.

We observe that the singular points $(x_i, 0)$, $i = 1, 2, 3$, where x_i are the solutions of the cubic equation $x^3 + cx + a = 0$ are located on the real invariant line $y = 0$. As the discriminant of this equation is also ξ, we deduce that all the real singularities are located on the real line $y = 0$ if $\xi \geq 0$ and there are one real and two complex singularities on this line if $\xi < 0$.

Thus we obtain the configuration corresponding to *Config. 8.10* if $\xi > 0$, *Config. 8.11* if $\xi < 0$ and *Config. 8.12* if $\xi = 0$ (see Figure 2.1).

Systems with configuration (3, 2, 1, 1). According to Theorem 2.2, if a cubic system possesses 7 invariant straight lines in the configuration $(3, 2, 1, 1)$, then necessarily the conditions $\mathcal{V}_4 = \mathcal{V}_5 = \mathcal{U}_2 = 0$ hold. So, as a first step, we need to force these conditions to be satisfied for systems (2.50). Using the mentioned conditions in [27, Section 3.7.1] we have determined the corresponding homogeneous cubic parts. More precisely, we have arrived to the following two homogeneous systems:

$$\dot{x} = 2x^3, \quad \dot{y} = 3x^2y + y^3 \tag{2.57}$$

and

$$\dot{x} = (1 + s^2)x^3, \quad \dot{y} = -sx^3 + s^2x^2y - sxy^2 - y^3. \tag{2.58}$$

On the other hand using the invariant polynomials we have distinguished the above systems as it is mentioned in the next remark.

Remark 2.5. *We note that for system* (2.57) *we have* $\mathcal{V}_3 = 0$*, whereas for systems* (2.58) *we have* $\mathcal{V}_3 = -32(9 + s^2)x^2(sx + y)^2 \neq 0$*. So for* $\mathcal{V}_4 = \mathcal{V}_5 = \mathcal{U}_2 = 0$ *we get system* (2.57) *if* $\mathcal{V}_3 = 0$ *and the family of systems* (2.58) *if* $\mathcal{V}_3 \neq 0$*. We also observe that for system* (2.57) *we have* $\mathcal{D}_4 = 0$*.*

Thus for the further examination it remains two families of systems with cubic homogeneities (2.57) and (2.58).

a) The family of systems with cubic homogeneities (2.57). For system (2.57) we calculate $H(X,Y,Z) = \gcd(\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3) = 6X^3Y(X^2+Y^2)^2$ and by Remark 2.2 the cubic systems with homogeneous parts (2.57) could possess one triplet of invariant lines only in the direction $x = 0$. As regard the two couples of parallel lines we conclude that they must be complex and in complex directions $y = \pm ix$. Therefore it is clear that in this case we could not have the configuration or potential configuration of the type (3, 2, 1, 1).

b) The family of systems with cubic homogeneities (2.58). For homogeneous cubic systems (2.58) we have

$$H(X,Y,Z) = (1+s^2)X^3(sX+Y)^2(X^2+Y^2). \tag{2.59}$$

Hence systems (2.58) possess one triple ($x = 0$) and one double ($sx+y = 0$) real lines as well as two complex invariant lines $y = \pm ix$. So by Remark 2.2 we conclude, that cubic systems

$$\begin{aligned} \dot{x} &= a + cx + dy + 2hxy + ky^2 + (1+s^2)x^3, \\ \dot{y} &= b + ex + fy + lx^2 + 2mxy - sx^3 + s^2x^2y - sxy^2 - y^3 \end{aligned} \tag{2.60}$$

with the cubic homogeneities (2.58) (here we assume $g = n = 0$ due to a rescaling) could have a triplet only in the direction $x = 0$ and a couple of parallel lines only in the direction $y = -sx$. Moreover these systems could have two simple complex conjugate invariant lines. Using the equations (2.17) we evaluate them for each one of these directions.

(i) The direction $x = 0$. In this case we obtain

$$Eq_7 = k, \quad Eq_9 = d - 2hW, \quad Eq_{10} = a - cW - (1+s^2)W^3 \tag{2.61}$$

and to have exactly three parallel invariant lines in this direction the condition $k = d = h = 0$ is necessary and sufficient.

(ii) The direction $sx + y = 0$. Then for the systems (2.60) with $k = d = h = 0$ we calculate

$$Eq_5 = l - 2ms, \; Eq_8 = e + s(c-f) - 2mW + 2sW^2, \; Eq_{10} = b + as - fW + W^3 \tag{2.62}$$

and to have exactly two parallel invariant lines in this direction the conditions $l = 2ms$ and $s \neq 0$ are necessary. Moreover in order to have two invariant lines in the direction $sx+y = 0$ the following additional conditions are necessary and sufficient: $R_W^{(0)}(Eq_8, Eq_{10}) = R_W^{(1)}(Eq_8, Eq_{10}) = 0$. Assuming $l = 2ms$ (then $Eq_5 = 0$) we calculate $R_W^{(1)}(Eq_8, Eq_{10}) = -2\big[es - 2m^2 + (c+f)s^2\big] = 0$ and as $s \neq 0$ we obtain $e = \big[2m^2 - (c+f)s^2\big]/s$. Then we calculate $R_W^{(0)}(Eq_8, Eq_{10}) = 8(fms^2 + bs^3 + as^4 - m^3)^2/s^3 = 0$, and this yields $b = -\big[fms^2 + as^4 - m^3\big]/s^3$.

(iii) The direction $x + iy = 0$. In this case we obtain the equations $Eq_5 = i(i+s)\big[2m + (3+is)W\big] = 0$, $Eq_8 = c - f + i(2m^2/s - cs - fs) - 2mW + (is-3)W^2 = 0$, $Eq_{10} = a + i(m^3/s^3 - fm/s - as) - fW - W^3 = 0$. So for the existence of an invariant line in this complex direction we must have $W = -2m/(3+is)$ and then we calculate

$$Eq_8 = (c-f) + \frac{48m^2s^2}{(9+s^2)^2} - i\Big[(c+f)s + \frac{6m^2(s^4-18s^2-27)}{s(9+s^2)^2}\Big],$$

$$Eq_{10} = a + \frac{6fm}{9+s^2} - \frac{72m^3(s^2-3)}{(9+s^2)^3} - i\Big[as + \frac{3fm(3+s^2)}{s(9+s^2)} - \frac{9m^3(s^2-3)(s^4-18s^2-27)}{s^3(9+s^2)^3}\Big].$$

Since the coefficients of cubic systems (2.60) are real, the conditions $Eq_8 = Eq_{10} = 0$ lead to the following four equalities:

$$(c-f) + \frac{48m^2s^2}{(9+s^2)^2} = (c+f)s + \frac{6m^2(s^4-18s^2-27)}{s(9+s^2)^2} = 0,$$
$$a + \frac{6fm}{9+s^2} - \frac{72m^3(s^2-3)}{(9+s^2)^3} = as + \frac{3fm(3+s^2)}{s(9+s^2)} - \frac{9m^3(s^2-3)(s^4-18s^2-27)}{s^3(9+s^2)^3} = 0.$$

Herein we obtain the following relations

$$c = -\frac{27m^2(s^2-3)(1+s^2)}{s^2(9+s^2)^2}, \quad f = \frac{3m^2(27+18s^2+7s^4)}{s^2(9+s^2)^2}, \quad a = -\frac{54m^3(1+s^2)}{s^2(9+s^2)^2}$$

and therefore we get the following dependencies among the parameters of systems (2.60):

$$\begin{aligned} &k = d = h = 0,\ l = 2ms,\ c = -\frac{27m^2(-3+s^2)(1+s^2)}{s^2(9+s^2)^2},\ e = \frac{8m^2s(s^2-9)}{(9+s^2)^2}\\ &f = \frac{3m^2(27+18s^2+7s^4)}{s^2(9+s^2)^2},\ a = -\frac{54m^3(1+s^2)}{s^2(9+s^2)^2},\ b = \frac{2m^3(9+17s^2)}{s(9+s^2)^2}. \end{aligned} \tag{2.63}$$

Thus we arrive to the family of systems

$$\begin{aligned} \dot{x} =& (1+s^2)\Big(x - \frac{6m}{9+s^2}\Big)\Big[x^2 + \frac{6m}{9+s^2}x + \frac{9m^2}{s^2(9+s^2)}\Big],\\ \dot{y} =& b + ex + fy + 2msx^2 + 2mxy - sx^3 + s^2x^2y - sxy^2 - y^3, \end{aligned} \tag{2.64}$$

where the parameters b, e and f have the values indicated above.

Assume first $m \neq 0$. Since $s \neq 0$ it is easy to find out that via the transformation $x_1 = \dfrac{s(9+s^2)}{9m}x - \dfrac{2s}{3}$, $y_1 = \dfrac{s(9+s^2)}{9m} + \dfrac{2s^2}{9}$, $t_1 = \dfrac{81m^2}{s^2(9+s^2)^2}t$ the above systems could be brought to the 1-parameter family of systems (we keep the old notations of variables)

$$\begin{aligned} \dot{x} =& (1+s^2)x\big[(x+s)^2 + 1\big],\\ \dot{y} =& (1+s^2)^2y + 2s(1+s^2)xy - sx^3 + s^2x^2y - sxy^2 - y^3. \end{aligned} \tag{2.65}$$

These systems possess the following invariant lines: $L_1 = x$, $L_2 = x + s + i$, $L_3 = x + s - i$, $L_4 = sx + y$, $L_5 = sx + y + 1 + s^2$, $L_6 = y + ix$, $L_7 = y - ix$ and it is clear that all these lines are distinct.

Systems (2.65) possess the following 3 real and 6 complex finite singularities: $(0, 0)$, $(0, \pm(1 + s^2))$, $\big(i - s, \pm(1 + is)\big)$, $\big(-i - s, \pm(1 - is)\big)$, $\big(i - s, s(s - i)\big)$, $\big(-i - s, s(s + i)\big)$. We observe that all singular points except $(0, 1 + s^2)$ are located at the intersections of the invariant lines and this leads to the configuration *Config. 8.13* (see Figure 1).

Assume now $m = 0$. Then systems (2.64) become the homogeneous systems (2.58) and considering (2.59) we deduce that these systems possess two real invariant straight lines $x = 0$ (triple) and $y + sx = 0$ (double), as well as two complex lines $y = \pm ix$. Therefore in this case we obtain the configuration given by *Config. 8.14* from Figure 2.1.

Systems with configuration (2, 2, 2, 1). Firstly, according to Theorem 2.2, for a system (2.50) we need to force the condition $\mathcal{V}_3 = 0$. In [27, Section 3.8.1] it was shown that in this case systems (2.50) have the homogeneous cubic part

$$\dot{x} = -2x^3 + 2sx^2y, \quad \dot{y} = -sx^3 - 3x^2y + sxy^2 - y^3. \tag{2.66}$$

Moreover, as due to a translation in the quadratic part of systems (S) we can consider $g = n = 0$, we arrive at the following family of systems:

$$\begin{aligned} \dot{x} =& a + cx + dy + 2hxy + ky^2 - 2x^3 + 2sx^2y, \\ \dot{y} =& b + ex + fy + lx^2 + 2mxy - sx^3 - 3x^2y + sxy^2 - y^3. \end{aligned} \tag{2.67}$$

Remark 2.6. *We remark that due to the change $y \to -y$ for* (2.67) *we may assume $s \geq 0$.*

For homogeneous cubic system (2.66) we have $H(X, Y, Z) = 2X^2(sX + Y)(X^2 + Y^2)^2$. So systems (2.66) possess three double invariant lines: the real line $x = 0$ and two complex invariant lines $y = \pm ix$. Hence by Remark 2.2 we conclude, that cubic systems (2.67) could have three couples of parallel lines only in these directions. Moreover these systems could have one simple real invariant line in the direction $y = -sx$.

(i) The direction $x = 0$. In this case we obtain the following non-vanishing equations: $Eq_7 = k$, $Eq_9 = d - 2hW + 2sW^2$, $Eq_{10} = a - cW + 2W^3$. Therefore to have exactly two parallel invariant lines in this direction it is necessary and sufficient $Eq_7 = 0$, $s \neq 0$ and $R^{(0)}_W(Eq_9, Eq_{10}) = R^{(1)}_W(Eq_9, Eq_{10}) = 0$. So $k = 0$ and then we calculate $R^{(1)}_W(Eq_9, Eq_{10}) = 4(2h^2 - ds - cs^2) = 0$. This yields $d = (2h^2 - cs^2)/s$ and we have $R^{(0)}_W(Eq_9, Eq_{10}) = 8(-2h^3 + chs^2 + as^3)^2/s^3 = 0$ which implies $a = h(2h^2 - cs^2)/s^3$.

(ii) The direction $x+iy=0$. We obtain $Eq_7=2m-i(l+2h)$, $Eq_{10}=(2h^3-chs^2)/s^3+ib-(c+ie)W+ilW^2+(2+is)W^3$, $Eq_9=(2h^2)/s-cs+e+i(f-c)-2(l+h+im)W+(3i-s)W^2$ and to have exactly two parallel invariant lines in this direction it is necessary $Eq_7=0$. As the parameters of cubic systems are real the equality $Eq_7=0$ gives $m=0$, $l=-2h$. So considering the relations

$$k=0,\quad d=(2h^2-cs^2)/s,\quad a=h(2h^2-cs^2)/s^3,\quad m=0,\quad l=-2h, \tag{2.68}$$

determined at this moment among the parameters of systems (2.67), we examine the fourth direction: $sx+y=0$.

(iii) The direction $sx+y=0$. Considering the conditions (2.68) we find out the next equations which do not vanish

$$\begin{gathered}Eq_5=(1+s^2)(3W-2h),\ Eq_8=e+(c-f-2h^2)s+cs^3-2hsW+4sW^2,\\ Eq_{10}=b-ch+2h^3/s^2-(f+2h^2-cs^2)W+W^3.\end{gathered} \tag{2.69}$$

Hence the unique value for the parameter W given by the equality $Eq_5=0$ is $W=2h/3$. Then calculations yield: $Eq_8\big|_{W=2h/3}=e+(c-f)s+cs^3-14h^2s/9=0$, $Eq_{10}\big|_{W=2h/3}=b-h(3c+2f-2cs^2)/3+2h^3(27-14s^2)/(27s^2)=0$ and we get $e=(f-c)s-cs^3+14h^2s/9$, $b=h\big[(27c+18f+28h^2)s^2-18cs^4-54h^2\big]/(27s^2)$. Therefore, considering the above mentioned equations for the direction $x+iy=0$ we obtain $Eq_9=2h^2(9+7s^2))/(9s)-2cs+fs-cs^3+i(f-c)+2hW+(3i-s)W^2$ and $Eq_{10}=h(2h^2-cs^2)/s^3-ih(54h^2-27cs^2-18fs^2-28h^2s^2+18cs^4)/(27s^2)-\big[c+i(fs-cs-cs^2+14h^2s/9\big]W-2ihW^2+(2+is)W^3$ and the conditions

$$R_W^{(0)}(Eq_9,Eq_{10})=R_W^{(1)}(Eq_9,Eq_{10})=0 \tag{2.70}$$

have to be satisfied. We calculate

$$R_W^{(1)}(Eq_9,Eq_{10})=\Phi_1(c,f,h,s)+i\Phi_2(c,f,h,s)=0,$$

where $\Phi_1=(3-s^2)(c+2f-cs^2)+2h^2(27-7s^2)/9$, $\Phi_2=8fs-4cs(s^2-1)+4h^2(5s^2-9)/(3s)$. Therefore the relation $\Phi_2=0$ gives $f=\Big[3cs^2(s^2-1)+h^2(9-5s^2)\Big]/(6c^2)$ and then we obtain $\Phi_1=h^2(s^2-9)^2/9s^2=0$. So we have either $s=\pm3$ or $h=0$ and in both cases the conditions (2.70) are fulfilled.

In the case $s=\pm3$ by Remark 2.6 we may assume $s=3$ and we get $f=2(6c-h^2)/3$. Then we obtain $\Phi_1(c,f,h,s)=\Phi_2(c,f,h,s)=0$ and $R_W^{(0)}(Eq_9,Eq_{10})=0$. In such a way we arrive at the following relations among the parameters of systems (2.67):

$$\begin{gathered}s=3,\ k=m=0,\ l=-2h,\ d=(2h^2-9c)/3,\ e=2(4h^2-27c)/3,\\ f=2(6c-h^2)/3,\ a=h(2h^2-9c)/27,\ b=h(10h^2-63c)/27\end{gathered} \tag{2.71}$$

and this leads to systems (for the further necessity we set here $h = h_1$ and $c = c_1$)

$$\begin{aligned}\dot{x} =& \big[(h_1+6x)^2+3(h_1^2-6c_1)\big](h_1-3x+9y)/54,\\ \dot{y} =& \frac{h_1}{27}(10h_1^2-63c)+\frac{2}{3}(4h_1^2-27c_1)x+\frac{2}{3}(6c_1-h_1^2)y-2h_1x^2-3x^3-3x^2y+3xy^2-y^3.\end{aligned}\tag{2.72}$$

In the case $h = 0$ we arrive at the following relations among the parameters of systems (2.67):

$$\begin{gathered}k = m = h = l = 0, \quad d = -cs, \quad e = -cs(3+s^2)/2,\\ f = c(s^2-1)/2, \quad a = b = 0\end{gathered}\tag{2.73}$$

and this leads to the following family of systems:

$$\begin{aligned}\dot{x} =& (c-2x^2)(x-sy),\\ \dot{y} =& -cs(3+s^2)x/2+c(s-1)y/2-sx^3-3x^2y+sxy^2-y^3.\end{aligned}\tag{2.74}$$

We observe that systems (2.72) (respectively systems (2.74)) possess two parallel invariant lines in the direction $x = 0$, which are real if $6c_1 - h_1^2 > 0$ (respectively $c > 0$); complex if $6c_1 - h_1^2 < 0$ (respectively $c < 0$) and they coincide if $6c_1 - h_1^2 = 0$ (respectively $c = 0$).

It is easy to check that in the case $(6c_1 - h_1^2)c \neq 0$ as well as in the case $6c_1 - h_1^2 = c = 0$ systems (2.72) could be brought to the systems (2.74) with $s = 3$ via the transformation $x_1 = \alpha x + h_1\alpha/6$, $y_1 = \alpha y + h_1\alpha/6$, $t_1 = t/\alpha^2$, where $\alpha = \sqrt{6c/(6c_1 - h_1^2)}$ if $(6c_1 - h_1^2)c > 0$ and $\alpha = 1$ if $6c_1 - h_1^2 = c = 0$.

Thus it was proved the next lemma.

Lemma 2.10. *A system* (2.67) *possesses the configuration or potential configuration of invariant lines* $(2, 2, 2, 1)$ *if and only this system via an affine transformation and time rescaling could be brought to a cubic system belonging to the subfamily* (2.74), *which is defined in the family* (2.67) *by the conditions*

$$s\neq 0,\ k=m=h=l=0,\ d=-cs,\ e=-cs(3+s^2)/2,\ f=c(s^2-1)/2,\ a=b=0.\tag{2.75}$$

Next we examine systems (2.74), considering each one of the cases: $c > 0$, $c < 0$ and $c = 0$.

1) The case $c > 0$. Then we may assume $c = 2u^2 \neq 0$ and via the transformation $\big(x, y, t\big) \mapsto \big(-(2x+1)u,\ (s-2y)u,\ t/(4u^2)\big)$ systems (2.74) can be brought to the systems

$$\begin{aligned}\dot{x} =& x(x-1)(1+s^2-2x+2sy),\\ \dot{y} =& -sx^3-y-s^2y+3xy+s^2xy-3x^2y-2sy^2+sxy^2-y^3.\end{aligned}\tag{2.76}$$

These systems possesses the invariant lines:

$$x = 0, \quad x = 1, \quad y = -sx, \quad y = \pm ix, \quad y \pm i(x-1) + s = 0$$

and considering its nine finite singularities: $(0,0)$, $(1,-s)$, $(1/2,-s/2)$, $(1,\pm i)$, $(0,-s\pm i)$, $\big((1+is)/2,(i-s)/2\big)$, $\big((1-is)/2,(-i-s)/2\big)$ we arrive at the configuration *Config. 8.15* (see Figure 2.1).

2) The case $c<0$. Then we may assume $c=-2u^2\neq 0$ and due to the rescaling $(x,y,t)\mapsto(ux,uy,t/u^2)$ we obtain the systems

$$\begin{aligned}\dot{x} =&2(1+x^2)(sy-x-s),\\ \dot{y} =&s(s^2+3)x+(1-s^2)y-sx^3-3x^2y+sxy^2-y^3.\end{aligned}\tag{2.77}$$

These systems possess the invariant lines $y=-sx$, $x=\pm i$, $y-ix\pm(1-is)=0$, $y+ix\pm(1+is)=0$ and the nine finite singularities: $(0,0)$, $(-s,-1)$, $(s,1)$, $\big(i,\,is\pm 2\big)$, $\big(-i,\,-is\pm 2\big)$, $\big(i,\,-is\big)$, $\big(-i,\,is\big)$. Therefore we arrive at the configuration *Config. 8.16* from Figure 2.1.

3) The case $c=0$. Then systems (2.74) become the homogeneous systems (2.66), which possess the real invariant lines $x=0$ (double) and $y=-sx$ (simple) as well as the complex invariant lines $y=\pm ix$ (both doubles). As a result we get the configuration of invariant lines given by *Config. 8.17* in Figure 2.1.

2.3. Invariant criteria for the realization of the configurations with four distinct infinite singularities

2.3.1. Conditions for *Config. 8.1–Config. 8.9*

According to Lemma 2.2 the conditions $\mathcal{D}_1>0$, $\mathcal{D}_2>0$, $\mathcal{D}_3>0$ are necessary and sufficient for a cubic systems to have four real distinct infinite singularities and via a linear transformation a cubic system could be brought to the form (2.18). Next we will prove the statements $A1), A2)$ and $A3)$ of the Main Theorem A which lead to the configurations *Config. 8.1 - 8.3*, *Config. 8.4 - 8.6* and *Config. 8.7 - 8.9*, respectively.

The statements $\mathbf{A_1}$). By Theorem 2.2 for the cubic systems with two triplets of parallel invariant lines the conditions $\mathcal{V}_1=\mathcal{V}_2=\mathcal{U}_1=0$ are satisfied and in this case a cubic system (2.18) via a linear transformation and a time rescaling could be brought to the form (2.21). Moreover, it was proved earlier in the previous subsection that systems (2.21) have the configurations *Config. 8.1 - 8.3* if and only if the conditions

$$k=d=h=l=e=m=c-f=a^2-b^2=0,\quad a^2+b^2\neq 0\tag{2.78}$$

are fulfilled. Since for these systems the conditions $\mathcal{V}_1=\mathcal{V}_2=\mathcal{U}_1=0$ hold, according to the statement $A_1)$ of the Main Theorem A it remains to prove that the conditions (2.78) are equivalent to $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{K}_1 = 0$ and $\mathcal{K}_2 \neq 0$ and to distinguish *Config. 8.1 - 8.3.*

For systems (2.21) we calculate $\mathcal{L}_1 = -2^8 3^4(lx^3 + 2mx^2y - 2hxy^2 - ky^3)$ and hence the condition $\mathcal{L}_1 = 0$ gives $l = m = h = k = 0$. Then we obtain $\mathcal{L}_1 = 0$ and $\mathcal{L}_2 = 2^7 3^5\big[-ex^2 - 6(c-f)xy + dy^2\big]$ and clearly the condition $\mathcal{L}_2 = 0$ implies $e = d = c - f = 0$ and this leads to the family of systems (2.24).

Next for systems (2.24) we calculate $\mathcal{K}_1 = 2^{18}3^{15}5^4 7^4 19 \cdot 41(a^2 - b^2)(x^2 - y^2)$, $\mathcal{K}_2 = -27x^4y^4(bx - ay)$. So clearly the condition $a^2 - b^2 = 0$ is equivalent to $\mathcal{K}_1 = 0$, whereas the condition $a^2 + b^2 \neq 0$ is equivalent to $\mathcal{K}_2 \neq 0$.

As it was mentioned earlier we could consider $b = a$ and then for systems (2.25) we need the expression $\text{Discrim}\,[a + cx + x^3, x] = -(27a^2 + 4c^3) = \xi$ which governs the type of the invariant lines (real, complex or coinciding) of these systems for which we calculate $\mathcal{K}_3 = -5400(27a^2 + 4c^3)x^4(x-y)^2y^4(x+y)^2(x^2+y^2)$. So clearly $27a^2 + 4c^3 = 0$ if and only if $\mathcal{K}_3 = 0$ and $\text{sign}\,(\mathcal{K}_3) = -\text{sign}\,(27a^2 + 4c^3) = \text{sign}\,(\xi)$.

To complete the proof of the statement A_1) of the Main Theorem A we construct the respective canonical systems corresponding to each of the configurations. We consider systems (2.25). Since the equation $z^3 + cz + a = 0$ possesses at least one real solution, say $z = z_0$ then applying the translation $x = x_1 + z_0$, $y = y_1 + z_0$ to the systems (2.25) we get the family of systems

$$\dot{x} = x(f + gx + x^2), \quad \dot{y} = y(f + gy + y^2). \tag{2.79}$$

a) Assume first $\xi > 0$. Then the systems above possess three distinct real lines in the direction $x = 0$ as well as three such lines in the direction $y = 0$. Therefore $g^2 - 4f > 0$ and setting $g^2 - 4f = u^2 > 0$ we obtain $f = (g^2 - u^2)/4$ where $g^2 - u^2 \neq 0$ because all the lines are distinct. Then via the rescaling $(x, y, t) \mapsto \big(2x/(g-u),\ 2y/(g-u),\ 4t/(g-u)^2\big)$ we obtain the following 1-parameter family of systems

$$\dot{x} = x(x+1)(x-a), \quad \dot{y} = y(y+1)(y-a), \tag{2.80}$$

where $a = (g+u)/(u-g)$. These systems possess the invariant lines $x = 0$, $x = -1$, $x = a$ $y = 0$, $y = -1$, $y = a$, $y = x$.

We claim that the parameter $a \neq 0$ could be considered positive and different from 1. Indeed suppose that $a < 0$. If $a < -1$ then via the transformation $(x, y, t) \mapsto (-x-1,\ -y-1, t)$ we obtain the systems $\dot{x} = x(x+1)(x-a')$, $\dot{y} = y(y+1)(y-a')$, where $a' = -(1+a) > 0$ as $a < -1$.

Assume now $-1 < a < 0$. Then applying the transformation $(x, y, t) \mapsto \big(a(x+1),\, a(y+1), t/a^2\big)$ we get the above systems with $a' = -(1+a)/a > 0$ as $-1 < a < 0$.

On the other hand considering the conditions provided by the statement A_1) of the Main Theorem A we calculate for systems (2.80): $\mathcal{K}_2 = (a-1)(2+a)(1+2a)x^4(x-y)y^4$. So the condition $\mathcal{K}_2 \neq 0$ implies $a \neq 1$. Therefore our claim is proved and for the canonical systems (2.80) we assume $a > 0$ and $a \neq 1$.

b) Admitting $\xi < 0$ we have $g^2 - 4f < 0$ and we can set $g^2 - 4f = -u^2 < 0$. Then $f = (g^2 + u^2)/4$ and after the additional rescaling $(x, y, t) \mapsto \big(ux/2,\ uy/2,\ 4t/u^2\big)$ we arrive at the systems

$$\dot{x} = x\big[(x+a)^2 + 1\big], \quad \dot{y} = y\big[(y+a)^2 + 1\big], \tag{2.81}$$

where $a = g/u$. We remark that these systems possess the invariant lines $x = 0$, $x = -a \pm i$, $y = 0$, $y = -a \pm i$, $y = x$. For these systems we have $\mathcal{K}_2 = 2a(9 + a^2)x^4(x-y)y^4$ and considering the condition $\mathcal{K}_2 \neq 0$ we obtain $a \neq 0$.

c) Suppose finally $\xi = 0$, i.e. the equation $z^3 + cz + a = 0$ possesses a real solution z_0 of the multiplicity at least two. Then applying the translation $x = x_1 + z_0$, $y = y_1 + z_0$ to the systems (2.25) we get the family of systems $\dot{x} = x^2(g+x)$, $\dot{y} = y^2(g+y)$. For these systems we calculate $\mathcal{K}_2 = -2g^3x^4(x-y)y^4$ and hence the condition $\mathcal{K}_2 \neq 0$ yields $g \neq 0$. Therefore via the rescaling $(x, y, t) \mapsto (gx,\, gy,\, t/g^2)$ we obtain the system

$$\dot{x} = x^2(1+x), \quad \dot{y} = y^2(1+y). \tag{2.82}$$

The statements $\mathbf{A_2}$). According to Lemma 2.8 for the existence of the configuration $(3, 2, 1, 1)$ the condition $(r-1)(2+r)(1+2r) \neq 0$ is necessary. On the other hand for systems (2.27) we have $\mathcal{D}_4 = -1152(r-1)(2+r)(1+2r)$ and hence the condition above is equivalent to $\mathcal{D}_4 \neq 0$.

Now we concentrate our attention on the conditions (2.35) and according to the statement A_2) of the Main Theorem A we prove that these conditions are equivalent to $\mathcal{K}_4 = \mathcal{K}_5 = \mathcal{K}_6 = 0$. For systems (2.27) we calculate

$$\mathcal{K}_4 = l(2+r)(1+2r)x^3/9 - 2h(-1+r)rx^2y/3 + 2(r-1)(-h+hr-3kr)xy^2/9 - kry^3$$

and due to the condition $r(r-1)(2+r)(1+2r) \neq 0$ clearly the condition $\mathcal{K}_4 = 0$ gives $k = l = h = 0$. Then we calculate

$$\mathcal{K}_5 = Z_1x^4 + Z_2x^3y + Z_3x^2y^2 + Z_4xy^3 + Z_5y^4,$$

where

$$
\begin{aligned}
Z_1 =& -30r\Big[3r(1-r)(c-f)+e(4+r+4r^2)-4m^2(r-1)\Big],\\
Z_2 =& -10\Big[54fr^2-27dr^2(r-1)+8e(r-1)(2+r)(1+2r)-18cr(1+r+r^2)-\\
&-8m^2(2+r)(1+2r)\Big]/3,\\
Z_3 =&10(r-1)\Big[5dr(r-1)+2c(1+7r+r^2)-2f(4+r+4r^2)-24m^2\Big],\\
Z_4 =&10\Big[6c(1+7r+r^2)-d(r-1)(4-53r+4r^2)-6f(4+r+4r^2)-72m^2\Big]/3,\\
Z_5 =&30d(1+7r+r^2).
\end{aligned}
$$

So we have the following relation: $Z_3-(r-1)Z_4 = 20d(r-1)^2(2-19r+2r^2)/3$. Therefore the conditions $Z_3 = Z_4 = Z_5 = 0$ imply $d = 0$ and then the relations $Z_1 = Z_2 = Z_3 = 0$ give:

$$
\begin{gathered}
c=-\frac{108m^2r(1+r+r^2)}{(r-1)^2(2+r)^2(1+2r)^2},\quad e=\frac{4m^2(r-1)(1+7r+r^2)}{(2+r)^2(1+2r)^2},\\
f=-\frac{12m^2(1+5r+15r^2+5r^3+r^4)}{(r-1)^2(2+r)^2(1+2r)^2}.
\end{gathered}
\tag{2.83}
$$

Thus we obtain the respective conditions from (2.35) and it remains to find out the invariant conditions corresponding to the expressions for the parameters a and b. For systems (2.27) with the conditions $k = l = h = 0$ and (2.83) we calculate: Coefficient$[\mathcal{K}_6, x^9y^2] = -6480(r-1)^2r^3\big[b(r-1)(2+r)^2(1+2r)^2+8m^3(1+7r+r^2)\big]/\big[(2+r)^2(1+2r)^2\big]$. So due to the condition $r(r-1)(2+r)(1+2r) \neq 0$ the condition $\mathcal{K}_6 = 0$ implies $b = -8m^3(1+7r+r^2)/[(r-1)(2+r)^2(1+2r)^2]$ and then we calculate: Coefficient$[\mathcal{K}_6, x^8y^3] = 7560(1-r)r^3\big[216m^3r-a(-1+r)^2(2+r)^2(1+2r)^2\big]/\big[(2+r)^2(1+2r)^2\big]$. Therefore the condition $\mathcal{K}_6 = 0$ implies $a = 216m^3r/\big[(r-1)^2(2+r)^2(1+2r)^2\big]$ and we arrive at the conditions (2.35). It remains to note that in this case $\mathcal{K}_6 = 0$.

On the other hand the conditions $u \neq 0$ and $u = 0$ (see the notation for the parameter u on the page 59) lead to different configurations of invariant lines for systems (2.36). So we need an invariant polynomial which govern this condition. For these systems we calculate $\mathcal{L}_1 = -6912u(r-1)r(2+r)(1+2r)x^2y$ and due to the condition $r(r-1)(2+r)(1+2r) \neq 0$ the condition $u = 0$ is equivalent to $\mathcal{L}_1 = 0$ and we get *Config. 8.6.*

As it was shown above in the case $u \neq 0$ we obtain the family of systems (2.37) which possess two distinct configurations (*Config. 8.4* and *Config. 8.5*) depending on the sign of the parameter r. On the other hand for systems (2.37) we have $\mathcal{K}_7 = 4r$ and hence this invariant polynomial distinguishes the mentioned configurations of invariant lines.
Thus the statement A_2) of the Main Theorem A is proved.

The statements $\mathbf{A_3}$). We showed earlier in the previous subsection that a system (2.39) with $h = 0$ possesses the configuration or potential configuration of invariant lines $(2, 2, 2, 1)$

if and only if the following conditions are satisfied:

$$\begin{gathered}(r-1)(r+2)(2r+1)\neq 0,\quad k=l=g=m=a=b=0,\\ d=f(2+r),\quad e=f(1+2r),\quad c=fr.\end{gathered}\tag{2.84}$$

Following the statement A_3) of the Main Theorem A we shall prove that these conditions are equivalent to $\mathcal{D}_4\neq 0,\quad \mathcal{V}_3=\mathcal{K}_4=\mathcal{K}_2=\mathcal{K}_8=0$. First of all we observe that for systems (2.39) with $h=0$, i.e. for systems

$$\begin{aligned}\dot{x}&=a+cx+dy+gx^2+ky^2+rx^3+(2+r)x^2y,\\ \dot{y}&=b+ex+fy+lx^2+2mxy+(1+2r)xy^2+y^3\end{aligned}\tag{2.85}$$

we have $\mathcal{D}_4=-1152(r-1)(2+r)(1+2r)$. Hence the first condition (2.84) is equivalent to $\mathcal{D}_4\neq 0$. For systems (2.85) we calculate $\mathcal{K}_4=\frac{2}{9}\big[l(r-1)(2+r)x^3+(2+r)(3l+g-2m+2gr-mr)x^2y+$
$+3(2m-kr+mr-2kr^2)xy^2+k(r-1)(1+2r)y^3\big]$ and it is obvious to detect that due to $(r-1)(r+2)(2r+1)\neq 0$ the condition $\mathcal{K}_4=0$ is equivalent to $k=l=g=m=0$.

Next we examine the conditions for the coefficients of linear terms given in (2.84). Considering the conditions above we calculate $\mathcal{K}_8=Z_1x^4+Z_2x^3y+Z_3x^2y^2+Z_4xy^3+Z_5y^4$, where

$$\begin{aligned}Z_1=&-5r^2(4c+e-f+8cr-4er-2fr),\\ Z_2=&r(16c+40e-40f+89cr-3dr-7er-83fr+6cr^2-6dr^2-60fr^2),\\ Z_3=&-16c-40e+40f-134cr+28dr-49er+89fr-89cr^2+49dr^2-28er^2+\\ &+134fr^2-40cr^3+40dr^3+16fr^3,\\ Z_4=&-60c-6e+6f-83cr-7dr-3er+89fr-40cr^2+40dr^2+16fr^2,\\ Z_5=&5(2c+4d-8f+cr-dr-4fr).\end{aligned}$$

It is not too difficult to detect, that the relations $Z_1=Z_2=Z_5=0$ yield $c=fr$, $d=f(2+r)$ and $e=f(1+2r)$ and then we get $\mathcal{K}_8=0$. Thus we obtain the respective conditions from (2.84) and it remains to find out the invariant conditions equivalent to $a=b=0$. We observe that for systems (2.85) in this case we have $\mathcal{K}_2=-3x^2y^2(rx+y)^2(bx-ay)(x+2rx+2y+ry)^2$ and evidently the condition $\mathcal{K}_2=0$ is equivalent to $a=b=0$.

In such a way we get the 2-parameter family of systems (2.49) possessing the configuration *Config. 8.7* if $f<0$; *Config. 8.8* if $f>0$ and *Config. 8.9* if $f=0$ (see Figure 2.1).

On the other hand for these systems we calculate $K_9=-180f(1+r)^2x^2y^2(rx+y)^2$ and as $r(r+1)\neq 0$ we conclude that $f=0$ if and only if $K_9=0$ and $\operatorname{sign}(K_9)=-\operatorname{sign}(f)$. This completes the proof of the statement A_3) of the Main Theorem A.

2.3.2. Conditions for *Config. 8.10–Config. 8.17*

According to Lemma 2.2 the conditions $\mathcal{D}_1 < 0$ are necessary and sufficient for a cubic systems to have two real and two complex all distinct infinite singularities and via a linear transformation a cubic system could be brought to the form (2.50). Next we will prove the statements A_4), A_5) and A_6) of the Main Theorem A which lead to the configurations *Config. 8.10 - 8.12*, *Config. 8.13, 8.14* and *Config. 8.15 - 8.17*, respectively.

The statements $\mathbf{A_4}$). It was shown earlier that for having the configurations of the type $(3,3,1)$ the conditions

$$k = h = l = m = e = d = c - f = ab = 0, \quad a^2 + b^2 \neq 0$$

must hold for systems (2.51). Following the statement A_4) of the Main Theorem A we prove that these conditions are equivalent to the affine invariant conditions

$$\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{U}_1 = \mathcal{L}_1 = \mathcal{L}_2 = \mathcal{K}_1 = 0, \ \mathcal{K}_2 \neq 0.$$

For systems (2.51) we calculate $\mathcal{L}_1 = 2^8 3^4 \big[(3l+2h)x^3+(k+6m)x^2y-(l+6h)xy^2-(3k+2m)y^3\big]$ and hence the condition $\mathcal{L}_1 = 0$ gives $l = h = k = m = 0$. Then we obtain $\mathcal{L}_1 = 0$, $\mathcal{L}_2 = 2^8 3^5 \big[(7d - 5e)x^2 - 2(c - f)xy + (5d - 7e)y^2\big]$ and clearly the condition $\mathcal{L}_2 = 0$ implies $e = d = c - f = 0$ and we arrive at the family of systems (2.53) with $e = 0$. So it remains to determine the invariant polynomials which govern the conditions $ab = 0$ and $a^2 + b^2 \neq 0$ for these systems. We calculate $\mathcal{K}_1 = 2^{32} 3^{15} 5^4 7^4 19 \cdot 43abxy$, $\mathcal{K}_2 = -27(bx - ay)(x^2 + y^2)^4$. So clearly the above mentioned condition $ab = 0$ is equivalent to $\mathcal{K}_1 = 0$ and $a^2 + b^2 \neq 0$ is equivalent to $\mathcal{K}_2 \neq 0$, respectively.

As it was mentioned earlier (see page 65) we could consider $b = 0$ and then for the corresponding systems we need the expression $\text{Discrim}\,[a + cz + z^3, z] = -(27a^2 + 4c^3) = \xi$ which governs the type of the invariant lines (distinct or coinciding) of these systems. We calculate $\mathcal{K}_3 = 2^{10} 3^3 5^2 (27a^2 + 4c^3)x^2y^2(x - y)^2(x^2 + y^2)^4$. So clearly $4a^2 + 27c^3 = 0$ if and only if $\mathcal{K}_3 = 0$ and $\text{sign}\,(\mathcal{K}_3) = -\text{sign}\,(27a^2 + 4c^3) = \text{sign}\,(\xi)$.

To complete the proof of the statement A_4) of the Main Theorem A it remains to construct the respective canonical systems for each one of the configurations. Since on the line $y = 0$ of systems (2.56) there exist a real solution x_0 of the cubic equation $x^3 + cx + a = 0$, then via the translation $(x, y) \mapsto (x + x_0, y)$ we get the family of systems

$$\dot{x} = c_1x + d_1x^2 - d_1y^2 + x^3 - 3xy^2, \quad \dot{y} = y(c_1 + 2d_1x + 3x^2 - y^2), \tag{2.86}$$

where $c_1 = c+3x_0^2$ and $d_1 = 3x_0$. Then on the invariant line $y = 0$ besides the singular point $(0, 0)$ there are locate two more singularities: $(x_{1,2}, 0)$, where $x_{1,2}$ are the solution of the quadratic equation $c_1 + d_1x + x^2 = 0$. The discriminant of this equation equals $\delta = d_1^2 - 4c_1$ and obviously we must have $\mathrm{sign}\,(\delta) = \mathrm{sign}\,(\xi)$ and $\delta = 0$ if and only if $\xi = 0$. For systems (2.86) we calculate $\mathcal{K}_2 = d_1(2d_1^2 - 9c_1)y(x^2 + y^2)^4$ and by the statement $A_4)$ of the Main Theorem A the condition $d_1 \neq 0$ must be satisfied.

a) Assume first $\xi \neq 0$. Then we can set $d_1^2 - 4c_1 = u^2\mathrm{sign}\,(\xi)$ and we obtain $c_1 = (d_1^2 - u^2\mathrm{sign}\,(\xi))/4$. Since $d_1 \neq 0$ and $\mathrm{sign}\,(\xi) = \mathrm{sign}\,(K_3)$ the systems (2.86) after the rescaling $(x, y, t) \mapsto (d_1x,\ d_1y, t/d_1^2)$ could be brought to the systems

$$\dot{x} = gx + x^2 - y^2 + x^3 - 3xy^2, \quad \dot{y} = gy + 2xy + 3x^2y - y^3, \tag{2.87}$$

where $g = [1 - a^2\mathrm{sign}\,(K_3)]/4 \neq 0$. So depending of the sign of the invariant polynomial K_3 we get the corresponding canonical systems given by the Main Theorem A (see the statement $A_4)$).

b) Suppose now $\xi = 0$. Then we have $c_1 = d_1^2/4$ and due to the same rescaling applied above we obtain systems (2.87) with $a = 0$.

The statements $\mathbf{A_5}$). According to Remark 2.5 the condition $\mathcal{V}_3 \neq 0$ distinguishes the systems (2.57) and (2.58). On the other hand for the existence of the configuration $(3, 2, 1, 1)$ the condition $s \neq 0$ is necessary. For systems (2.60) we have $\mathcal{D}_4 = 2304s(9+s^2)$ and hence, the above condition could be substituted by $\mathcal{D}_4 \neq 0$.

Now we concentrate our attention on the conditions (2.63). Following the statement $A_5)$ of the Main Theorem A we prove that these conditions for systems (2.60) are equivalent to the affine invariant conditions $\mathcal{K}_4 = \mathcal{K}_5 = \mathcal{K}_6 = 0$.

First we claim that the condition $\mathcal{K}_4 = 0$ is equivalent to $k = h = 0$ and $l = 2ms$. Indeed, for systems (2.60) we calculate Coefficient$[\mathcal{K}_4,\ y^3] = k(s^2 + 1)$ and clearly the condition $\mathcal{K}_4 = 0$ implies $k = 0$. Then we have Coefficient$[\mathcal{K}_4,\ xy^2] = 8hs^2/9 = 0$, i.e. $h = 0$ and in this case we calculate $\mathcal{K}_4 = (2ms - l)(9 + s^2)x^3/9 = 0$ and this implies $l = 2ms$. So our claim is proved. So it remains to prove that for the family of systems

$$\begin{aligned} \dot{x} &= a + cx + dy + (1 + s^2)x^3, \\ \dot{y} &= b + ex + fy + 2msx^2 + 2mxy + ny^2 - sx^3 + s^2x^2y - sxy^2 - y^3 \end{aligned} \tag{2.88}$$

the conditions

$$\begin{aligned} &d = 0,\ c = -\frac{27m^2(-3 + s^2)(1 + s^2)}{s^2(9 + s^2)^2},\ e = \frac{8m^2s(s^2 - 9)}{(9 + s^2)^2}, \\ &f = \frac{3m^2(27 + 18s^2 + 7s^4)}{s^2(9 + s^2)^2},\ a = -\frac{54m^3(1 + s^2)}{s^2(9 + s^2)^2},\ b = \frac{2m^3(9 + 17s^2)}{s(9 + s^2)^2}. \end{aligned} \tag{2.89}$$

are equivalent to the conditions $\mathcal{K}_5 = \mathcal{K}_6 = 0$.

Indeed, if for the above systems the conditions (2.89) are fulfilled then $\mathcal{K}_5 = \mathcal{K}_6 = 0$. Conversely, assume that $\mathcal{K}_5 = \mathcal{K}_6 = 0$ for systems (2.88). Calculations yield Coefficient$[\mathcal{K}_5,\ y^4] = 30d(9+5s^2) = 0$ and this implies $d = 0$. Then we obtain $\mathcal{K}_5 = 10Z_1x^4/3 + 20Z_2x^3y/3 + 20Z_3xy^2(sx+y)$, where

$$\begin{aligned}
Z_1 =& cs(81+180s^2-5s^4) + fs(5s^4-108s^2-81) + e(81+162s^2-47s^4) + 136m^2s^3,\\
Z_2 =& c(2s^4+27s^2-27) + f(27+9s^2-2s^4) + 8es(9+s^2) + 4m^2(8s^2-9),\\
Z_3 =& c(9+5s^2) + f(7s^2-9) - 12m^2.
\end{aligned}$$

Solving the system of equations $Z_1 = Z_2 = Z_3 = 0$ with respect to the parameters c, f and e we get the respective expressions from (2.89). Considering these values of the parameters we calculate Coefficient$[\mathcal{K}_6,\ x^5y^6] = 40(9+5s^2)^2(225+493s^2)\big(as^2(9+s^2)^2+54m^3(1+s^2)\big)/[9s^2(9+s^2)^2] = 0$ and this implies $a = -54m^3(1+s^2)/[s^2(9+s^2)^2]$. Then we obtain

$$\begin{aligned}
\mathcal{K}_6 = & -\frac{40}{9s(9+s^2)^2}\big[bs(9+s^2)^2 - 2m^3(9+17s^2)\big]x^6(sx+y)^2\big[(s^2-3)sx - (9+5s^2)y\big]\times\\
& \big[s^2(2781+5718s^2+409s^4)x^2 + 4s(9+s^2)(125s^2-33)xy + (9+5s^2)(407s^2-225)y^2\big]
\end{aligned}$$

and we observe that the condition $\mathcal{K}_6 = 0$ implies $b = 2m^3(9+17s^2)/[s(9+s^2)^2]$. So we get for the parameters a and b the expressions given in (2.89) and this completes the proof of the fact, that the conditions (2.89) are equivalent to the conditions $\mathcal{K}_5 = \mathcal{K}_6 = 0$.

It remains to find out the invariant polynomial which governs the condition $m = 0$ for systems (2.64). For these systems we calculate $\mathcal{L}_1 = 41472m(1+s^2)x^2(sx+y)$ and it is clear that the condition $\mathcal{L}_1 = 0$ is equivalent to $m = 0$. Thus the statement A_5) of the Main Theorem A is proved.

The statements $\mathbf{A_6}$). Considering the statement A_6) of the Main Theorem A and Lemma 2.10 we prove that the affine invariant conditions

$$\mathcal{D}_4 \neq 0, \quad \mathcal{V}_3 = \mathcal{K}_4 = \mathcal{K}_2 = \mathcal{K}_8 = 0 \tag{2.90}$$

applied to a cubic system (2.67) force this system to be from the class determined by Lemma 2.10. Indeed, for the family of systems (2.67) we have $\mathcal{D}_4 = 2304s(9+s^2)$ and hence the condition $s \neq 0$ is equivalent to $\mathcal{D}_4 \neq 0$. Assume that for a system (2.67) the condition $\mathcal{K}_4 = 0$ is satisfied. Then we obtain Coefficient$[\mathcal{K}_4, y^3] = -2k(9+s^2)/9$ and hence the condition $\mathcal{K}_4 = 0$ implies $k = 0$. In this case we obtain $\mathcal{K}_4 = \frac{2}{9}\big[(6ms-9h-2ls^2-3hs^2)x^3 - 2s(3l+6h+2ms)x^2y - (9h+6ms-hs^2)xy^2\big]$ and equalizing with zero the first two coefficients of the polynomial $\mathcal{K}_4$ due to $s \neq 0$ we get

$$l = -\frac{3h(9+s^2)}{9+2s^2}, \quad m = -\frac{3h(s^2-9)}{2s(9+2s^2)}, \quad \mathcal{K}_4 = -\frac{4h(s^2-9)(s^2+9)xy^2}{9(9+2s^2)}. \tag{2.91}$$

Therefore the condition $\mathcal{K}_4 = 0$ gives either $h = 0$, or $s = \pm 3$.

1) The case $h = 0$. Then we obtain $k = l = m = h = 0$ and we calculate

$$\mathcal{K}_2 = -3x^2(bx - ay)(3x + s^2x - 2sy)^2(x^2 + y^2)^2,$$
$$\mathcal{K}_8 = Z_1x^4 + Z_2x^3y + Z_3x^2y^2 + 3Z_4xy^3 + 5Z_5y^4,$$

where $Z_1 = -2(99d - 63e - 9fs + 15ds^2 - 10es^2 - 32cs^3 - 10fs^3)$, $Z_2 = 2(3c + 6f + 159ds - 103es - 3fs^2 + 50ds^3)$, $Z_3 = 2(54d + 27e + 105cs + 21fs + 45ds^2 - 20es^2 + 8cs^3 - 20fs^3)$, $Z_4 = 6(-9c - 18f + 3ds - es + 10cs^2 - fs^2)$, $Z_5 = 10(-3d - cs + 4fs + 2ds^2)$.

So setting $Z_2 = Z_4 = Z_5 = 0$ we find out that $d = -cs$, $e = -cs(3+s^2)/2$, $f = c(s^2-1)/2$ and then $\mathcal{K}_8 = 0$. On the other hand the condition $\mathcal{K}_2 = 0$ obviously gives $a = b = 0$ and hence in this case we arrive at the conditions (2.75).

2) The case $s = \pm 3$. By Remark 2.6 we may assume $s = 3$. Then $\mathcal{K}_4 = 0$ and we calculate

$$\mathcal{K}_8 = Z_1'x^4 + Z_2'x^3y + Z_3'x^2y^2 + 3Z_4'xy^3 + 5Z_5'y^4,$$

where $Z_1' = 18(96c - 26d + 17e + 33f - 6h^2)$, $Z_2' = 6(c + 609d - 103e - 7f - 136h^2)$, $Z_3' = -18(59c + 51d - 17e - 53f - 24h^2)$, $Z_4' = 6(81c - 4l^2 + 9d - 3e - 27f)$, $Z_5' = -10(3c - 15d - 12f + 2h^2)$. Setting $Z_2' = Z_4' = Z_5' = 0$ we obtain $d = (2h^2 - 9c)/3$, $e = 2(4h^2 - 27c)/3$, $f = 2(6c - h^2)/3$ and then $\mathcal{K}_8 = 0$ and $\mathcal{K}_2 = -4x^2(2x - y)^2(x^2 + y^2)^2\big[(27b + 63ch - 10h^3)x + (-27a - 9ch + 2h^3)y\big]$. Therefore the condition $\mathcal{K}_2 = 0$ yields $a = h(2h^2 - 9c)/27$, $b = h(10h^2 - 63c)/27$ and we arrive at the conditions (2.71) corresponding to the case $s = 3$. These conditions lead to the systems (2.72), which via an affine transformation and time rescaling could be brought to systems (2.74) as it was shown on the page 70.

Next we consider the necessary and sufficient conditions to distinguish the configurations *Config. 8.15* – *Config. 8.17*. For systems (2.74) we calculate $K_9 = 90c(1 + s^2)^2x^2(x^2 + y^2)^2$. Therefore we obtain that $K_9 = 0$ if and only if $c = 0$. Moreover if $K_9 \neq 0$ then $\text{sign}\,(K_9) = \text{sign}\,(c)$. Thus we get *Config. 8.15* if $K_9 > 0$, *Config. 8.16* if $K_9 < 0$ and *Config. 8.17* if $K_9 = 0$. This completes the proof of the statement A_6) of the Main Theorem A.

2.4. Conclusions on Chapter 2

In Chapter 2 we give 17 configurations (*Config. 8.1* – *Config. 8.17*, see Figure 2.1) of invariant lines of systems in $\mathbb{CSL}_8$ possessing 4 distinct infinite singularities.

On the other hand in [127] and [128] the same 17 configurations are also obtained by A. Şubă, V.Repeşco, V.Puţuntică with a minor difference. Namely, in [127] and [128] the authors do not include in the definition of configuration of invariant lines the real singularities located on the invariant lines. More precisely, in these articles cubic systems with exactly 7 invariant affine lines considered with their parallel multiplicity are examined. The authors say that an invariant line $f(x, y) = 0$ where $f(x, y) = ux + vy + w$ of a cubic system (2.1) has parallel multiplicity $1 \leq k \leq 3$ if the identity $\mathbb{X}(f) = f^k R(x, y)$ holds for some polynomial $R(x, y)$ with coefficients in $\mathbb{C}$. Taking into account the line at infinity, in fact the authors considered systems in $\mathbb{CSL}_8$. This coincidence is a natural one. Indeed, in our thesis we consider the class of cubic systems in $\mathbb{CSL}_8$ having four distinct $\mathbb{ISP}s$. And clearly a system in such a family could not have $\mathbb{IL}$ of, say, "non-parallel" multiplicity, because for these lines we must have at least one infinite singular point defined by a multiple linear factor of the form $C_3(x, y) = yp_3(x, y) - xq_3(x, y)$ when we factorize C_3 over $\mathbb{C}$. The cases of singularities at infinity defined by multiple factors of C_3 are considered in Chapters 3 and 4. We underline that in contrast to the results obtained in the papers [127] and [128] in this chapter we also find out the invariant criteria for the realization of each one of the 17 configurations.

The results exhibited in Chapter 2 were published in [20, 27].

3. CUBIC SYSTEMS WITH INVARIANT LINES OF TOTAL MULTIPLICITY EIGHT AND EITHER THREE DISTINCT OR ONE INFINITE SINGULARITIES

According to Lemma 2.2 we split the family of cubic systems having 3 distinct infinite singularities in two subfamilies: 1) systems with one double and two simple all real infinite singularities and 2) systems with one double real and two simple complex infinite singularities. In this chapter we also consider the subfamily of systems possessing exactly one infinite singularity. For each one of these three subfamilies the proofs of the corresponding theorems proceed in 4 steps described in Paragraph 2.1.2.

It is clear that if for perturbed systems some condition $K(x, y) = 0$ holds, where $K(x, y)$ is an invariant polynomial, then this condition must hold also for the initial (not-perturbed) systems. So considering the Corollary from the Main Theorem A we arrive at the next remark.

Remark 3.1. *Assume that a cubic system with at most three distinct infinite singularities possesses a potential configuration of a given type. Then for this system the following conditions must be satisfied, respectively:*

$$\begin{array}{lll} (a_1) & (3,3,1) & \Rightarrow \quad \mathcal{V}_1 = \mathcal{V}_2 = \mathcal{L}_1 = \mathcal{L}_2 = \mathcal{K}_1 = 0; \\ (a_2) & (3,2,1,1) & \Rightarrow \quad \mathcal{V}_5 = \mathcal{U}_2 = \mathcal{K}_4 = \mathcal{K}_5 = \mathcal{K}_6 = 0; \\ (a_3) & (2,2,2,1) & \Rightarrow \quad \mathcal{V}_3 = \mathcal{K}_4 = \mathcal{K}_2 = \mathcal{K}_8 = 0. \end{array}$$

In this chapter we prove the following two theorems:

Main Theorem B. *Assume that a non-degenerate cubic system (i.e. $\sum_{i=0}^{9} \mu_i^2 \neq 0$) possesses invariant straight lines of total multiplicity 8, including the line at infinity with its own multiplicity. In addition we assume that this system has three distinct infinite singularities, i.e. the conditions $\mathcal{D}_1 = 0$ and $\mathcal{D}_3 \neq 0$ hold. Then:*

I. *This system has only real infinite singularities and it possesses one of the five possible configurations Figure 3.1, endowed with the corresponding conditions included below. Moreover the system could be brought via an affine transformation and time rescaling to the canonical forms, written below next to the configurations.*

II. *This system could not have a configuration of invariant lines of of the type $(3, 3, 1)$ or $(3, 2, 2)$. And this system has:*

B_1) *Configuration of type* $(3, 2, 1, 1)$ $\Leftrightarrow \mathcal{V}_4 = \mathcal{V}_5 = \mathcal{K}_4 = \mathcal{K}_5 = \mathcal{K}_6 = 0$:

- *Config. 8.18* $\Leftrightarrow$ $\mathcal{K}_7 \neq 0$, $\mathcal{L}_1 \neq 0$: $\begin{cases} \dot{x} = x(x^2 - 9x - xy - y^2), \\ \dot{y} = -y^2(9 + y); \end{cases}$

- *Config. 8.19* $\Leftrightarrow$ $\mathcal{K}_7 \neq 0,\, \mathcal{L}_1 = 0$: $\begin{cases} \dot{x} = x(x^2 - xy - y^2), \\ \dot{y} = -y^3; \end{cases}$

- *Config. 8.20* $\Leftrightarrow$ $\begin{matrix} \mathcal{K}_7 = 0,\, \mathcal{L}_1 \neq 0, \\ \mathcal{V}_5 = \mathcal{L}_6 = 0 \end{matrix}$: $\begin{cases} \dot{x} = (1-x)x(1+y), \\ \dot{y} = y(1 - x + y - x^2). \end{cases}$

$B_2)$ *Configuration of type* $(2,2,2,1)$ $\Leftrightarrow \mathcal{V}_3 = \mathcal{K}_4 = \mathcal{K}_2 = \mathcal{K}_8 = 0, \quad \mathcal{L}_7 \neq 0:$

- *Config. 8.21* $\Leftrightarrow$ $\mathcal{K}_9 > 0$: $\begin{cases} \dot{x} = (x^2 - 1)(x + y), \\ \dot{y} = 2x(y^2 - 1); \end{cases}$

- *Config. 8.22* $\Leftrightarrow$ $\mathcal{K}_9 < 0$: $\begin{cases} \dot{x} = (1 + x^2)(x + y), \\ \dot{y} = 2x(1 + y^2). \end{cases}$

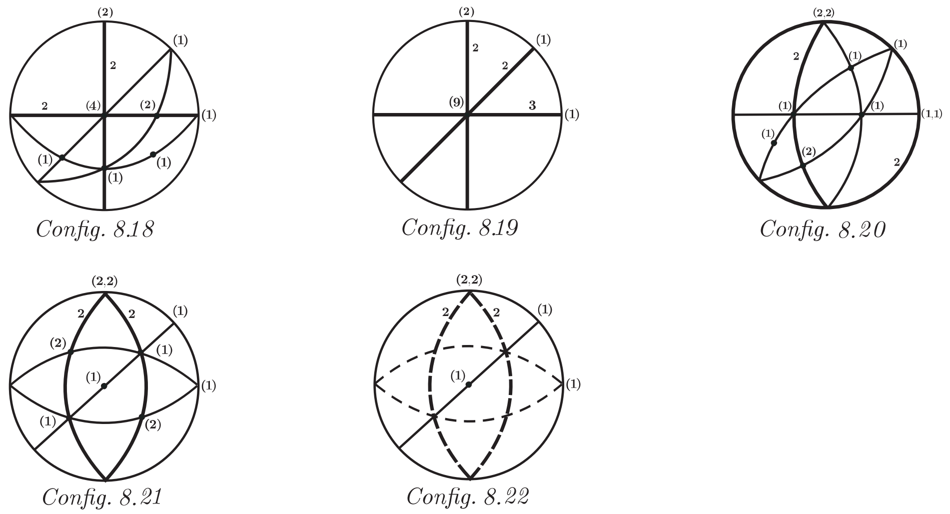

Fig. 3.1. Configurations of invariant lines for systems in $\mathbb{CSL}_8$ with 3 $\mathbb{ISP}s$

Main Theorem C. *Assume that a non-degenerate cubic system (i.e.* $\sum_{i=0}^{9} \mu_i^2 \neq 0$*) possesses invariant straight lines of total multiplicity 8, including the line at infinity with its own multiplicity. In addition we assume that this system has exactly one infinite singularity defined by a unique real factor of degree four of* $C_3(x,y)$*, i.e. the conditions* $\mathcal{D}_1 = \mathcal{D}_2 = \mathcal{D}_3 = 0$ *hold. Then this system possesses the specific configuration Config. 8.j* ($j \in \{48, \ldots, 51\}$) *(see Figure 3.2) if and only if the corresponding conditions included below are fulfilled. Moreover it can be brought via an affine transformation and time rescaling to the canonical form, written below next to the conditions:*

Config. 8.48 $\Leftrightarrow$ $\begin{bmatrix} \mathcal{V}_1 = \mathcal{L}_2 = N_{23} = W_1 = W_2 = \\ = W_3 = W_4 = 0 \end{bmatrix}$ $\Leftrightarrow$ $\begin{cases} \dot{x} = x, \\ \dot{y} = -2y - x^3; \end{cases}$

Config. 8.49 $\Leftrightarrow$ $\begin{bmatrix} \mathcal{V}_1 = \mathcal{L}_2 = N_{23} = W_1 = W_2 = \\ = N_{16} = W_5 = 0,\ W_6 \neq 0 \end{bmatrix}$ $\Leftrightarrow$ $\begin{cases} \dot{x} = x, \\ \dot{y} = y - x^2 - x^3; \end{cases}$

$$\textit{Config. 8.50} \Leftrightarrow \begin{bmatrix} \mathcal{V}_1 = \mathcal{L}_2 = N_3 = W_7 = W_8 = \\ = W_9 = W_{10} = 0,\ N_{23} \neq 0 \end{bmatrix} \Leftrightarrow \begin{cases} \dot{x} = x(1+x), \\ \dot{y} = y + xy - x^3; \end{cases}$$

$$\textit{Config. 8.51} \Leftrightarrow \begin{bmatrix} \mathcal{V}_5 = \mathcal{K}_4 = \mathcal{K}_5 = \mathcal{K}_8 = \mathcal{K}_9 = \\ = N_2 = \mathcal{K}_6 = W_{11} = W_{12} = 0, \\ \mathcal{V}_1 \neq 0 \end{bmatrix} \Leftrightarrow \begin{cases} \dot{x} = x^2(1+x), \\ \dot{y} = -1-3x+x^2y-x^3. \end{cases}$$

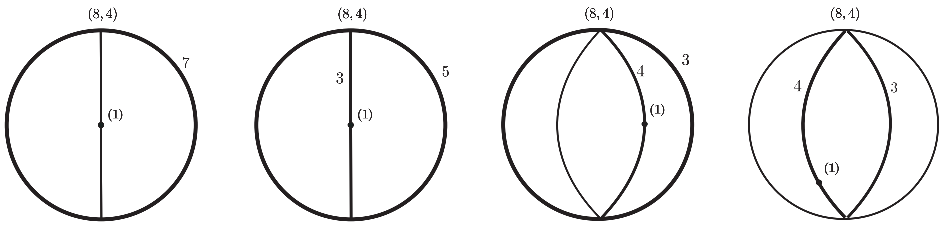

Fig. 3.2. Configurations of invariant lines for systems in $\mathbb{CSL}_8$ with one $\mathbb{ISP}$

The proofs of the above theorems are organized as follows. In Subsections 3.1 we consider the family of systems with three singularities at infinity. Since in [28] we have proved that systems with one real and two complex infinite singularities could not belong to $\mathbb{CSL}_8$, then in this subsection we consider only the subfamily of systems with three real infinite singularities. So we prove here Main Theorem B following all the steps described in Paragraph 2.1.2.

In Subsections 3.2 in the same manner we examine the family of systems with exactly one (real) infinite singularity proving Main Theorem C.

3.1. Cubic systems with three distinct infinite singularities

Assuming that systems (2.2) possess three real distinct infinite singularities (i.e. the conditions $\mathcal{D}_1 = 0$, $\mathcal{D}_3 > 0$ hold), according to Lemma 2.2 via a linear transformation they could be brought to the family of systems

$$\begin{aligned} \dot{x} &= a + cx + dy + gx^2 + 2hxy + ky^2 + (u+1)x^3 + (v-1)x^2y + rxy^2, \\ \dot{y} &= b + ex + fy + lx^2 + 2mxy + ny^2 + ux^2y + vxy^2 + ry^3, \quad C_3 = x^2y(x-y). \end{aligned} \tag{3.1}$$

In what follows for the above systems we construct step by step their canonical forms and corresponding configurations of invariant lines which could be of the types mentioned in Remark 3.1.

3.1.1. Construction of normal forms and of the corresponding configurations of invariant lines

We note that due to the existence of 3 distinct infinite singularities, only the configuration (3, 3, 1) could have 8 distinct invariant straight lines, whereas the configurations of other type could be only potential configurations for systems (3.1) (see the respective definition on p. 49).

Systems with configuration $(\mathbf{3},\,\mathbf{3},\,\mathbf{1})$. Since we have two triplets of parallel invariant straight lines, according to Theorem 2.2 the condition $\mathcal{V}_1 = \mathcal{V}_2 = 0$ is necessary for systems (3.1). For these systems we calculate: $\mathcal{V}_1 = 16\sum_{j=0}^{4}\mathcal{V}_{1j}x^{4-j}y^j$, $\mathcal{V}_2 = 8\sum_{j=0}^{2}\mathcal{V}_{2j}x^{2-j}y^j$, where

$$\begin{aligned}&\mathcal{V}_{10} = u(3+2u),\ \mathcal{V}_{11} = -2u+4uv+3v,\ \mathcal{V}_{12} = -1-2v+4ru+3r+2v^2,\\ &\mathcal{V}_{13} = 2r(2v-1),\ \mathcal{V}_{14} = 2r^2,\ \mathcal{V}_{20} = -3v-2u,\ \mathcal{V}_{21} = 6r+4v-2,\ \mathcal{V}_{22} = -2r.\end{aligned}$$

Consequently, the condition $\mathcal{V}_{14} = 2r^2 = 0$ implies $r = 0$ and then we obtain the following contradictory relations: $\mathcal{V}_{10} = u(2u+3) = 0$, $4\mathcal{V}_{20}+3\mathcal{V}_{21} = -2(4u+3) = 0$. So the conditions $\mathcal{V}_1 = \mathcal{V}_2 = 0$ cannot be satisfied for systems (3.1).

Systems with potential configuration $(\mathbf{3},\mathbf{2},\mathbf{1},\mathbf{1})$. First we construct the corresponding cubic homogeneities for systems (3.1). According to Theorem 2.2, if a cubic system possesses 7 invariant straight lines in the configuration $(3,2,1,1)$, then necessarily the conditions $\mathcal{V}_4 = \mathcal{V}_5 = \mathcal{U}_2 = 0$ hold. So we consider the homogeneous cubic part of systems (3.1)

$$\dot{x} = (u+1)x^3 + (v-1)x^2y + rxy^2, \quad \dot{y} = ux^2y + vxy^2 + ry^3 \tag{3.2}$$

and we force the conditions $\mathcal{V}_4 = \mathcal{V}_5 = \mathcal{U}_2 = 0$ to be satisfied. For systems (3.2) we have $\mathcal{V}_5 = \dfrac{9}{32}\sum_{j=0}^{4}\mathcal{V}_{5j}x^{4-j}y^j$, where $\mathcal{V}_{52} = 6r^2u$, $\quad\mathcal{V}_{54} = -r^2(1+r+v)$.

1) *The case $r \neq 0$.* Then $u = 0$, $v = -(r+1)$ and we get $\mathcal{V}_4 = 18432(r^2-1)x^2y(x-y) = 0$ and $\mathcal{U}_2 = -12288(r^2-1)(x-y)y(6x^2+r^2xy-r^2y^2) = 0$. So the condition $\mathcal{V}_4 = 0$ implies $\mathcal{U}_2 = 0$ (and in this case $\mathcal{V}_5 = 0$). Thus in the case $r = -1$ we get the system

$$\dot{x} = x^3 - x^2y - xy^2, \quad \dot{y} = -y^3. \tag{3.3}$$

In the case $r = 1$ we obtain the system which via the transformation $(x,y,t) \mapsto (x, x-y, -t)$ can be brought to system (3.3).

2) *The case $r = 0$.* We determine $\mathcal{V}_{50} = u(-2+v)(1+u+v) = 0$, $\mathcal{V}_4 = 9216(v-1)vx^2y(y-x) = 0$ and $\mathcal{U}_2 = 12288(1-v)x^2\big[u(uv-3-3u)x^2 + v(2uv-3-4u)xy + v(1-v+v^2)y^2\big]$ and therefore we get either $v = 0$ and $u(u+1) = 0$ or $v = 1$ and $u(u+2) = 0$. We note that in both cases the condition $\mathcal{V}_5 = \mathcal{V}_4 = 0$ holds and this implies $\mathcal{U}_2 = 0$.

a) *The subcase $v = 0$.* Then for $u = -1$ we get the system

$$\dot{x} = -x^2y, \quad \dot{y} = -x^2y. \tag{3.4}$$

In the case $u = 0$ we arrive at a system which can be brought to system (3.4) due to the transformation $(x,y,t) \mapsto (x, x-y, -t)$.

b) *The subcase* $v = 1$. Since the additional condition $u(u+2) = 0$ holds, we obtain the systeme

$$\dot{x} = x^3, \qquad \dot{y} = xy^2 \tag{3.5}$$

in the case $u = 0$. If $u = -2$ we get systems which due to the transformation $(x, y, t) \mapsto (x, x - y, -t)$ could be brought to system (3.5).

Thus for the further investigation we will use three different homogeneous systems: (3.3), (3.4) and (3.5). We observe that for system (3.3) we have $\mathcal{K}_7 = -4 \neq 0$, whereas for systems (3.4) and (3.5) the condition $\mathcal{K}_7 = 0$ holds. Moreover for system (3.4) we have $\mathcal{V}_6 = 0$ and for system (3.5) we have $\mathcal{V}_6 = 192x^3y \neq 0$ On the other hand as it was shown above the condition $\mathcal{V}_4 = \mathcal{V}_5 = 0$ implies $\mathcal{U}_2 = 0$. So we arrive at the following remark.

Remark 3.2. *If* $\mathcal{V}_4 = \mathcal{V}_5 = 0$ *then systems* (3.2) *due to a linear transformation can be brought to system* (3.3) *if* $\mathcal{K}_7 \neq 0$*; to system* (3.4) *if* $\mathcal{K}_7 = \mathcal{V}_6 = 0$ *and to system* (3.5) *if* $\mathcal{K}_7 = 0$ *and* $\mathcal{V}_6 \neq 0$.

Next step consists of the construction of the canonical systems which possess the required configuration. We shall examine each one of the cases, when the homogeneous cubic part of systems (3.1) corresponds either to system (3.3) or (3.4) or (3.5).

A) Systems with cubic homogeneous parts (3.3)**.** In this case due to a translation we may assume that in the quadratic part of the cubic systems the condition $g = n = 0$ holds. So we consider the family of systems

$$\begin{aligned} \dot{x} =& a + cx + dy + 2hxy + ky^2 + x^3 - x^2y - xy^2, \\ \dot{y} =& b + ex + fy + lx^2 + 2mxy - y^3. \end{aligned} \tag{3.6}$$

In what follows we shall determine necessary and sufficient conditions for a system (3.6) to have a potential configuration of invariant lines of type (3, 2, 1, 1).

Taking into consideration Remark 3.1 we observe that in this case for systems (3.6) the conditions $\mathcal{K}_4 = \mathcal{K}_5 = \mathcal{K}_6 = 0$ must be fulfilled. We calculate $\mathcal{K}_4 = -lx^3 + (7l - 8m)x^2y/9 + (l + 4m)xy^2/9 + (2h + k + 2m)y^3/9$ and therefore the condition $\mathcal{K}_4 = 0$ gives $l = m = 0$ and $k = -2h$. Then we obtain Coefficient$[\mathcal{K}_5, x^4] = 150e = 0$, i.e. $e = 0$ and we get $\mathcal{K}_5 = 20(7c + 5f + 12h^2)x^2y(x - y) + 40(c + 4d - f + 16h^2)xy^3/3 - 10(5c + 47d - 5f - 136h^2)y^4/3$. In this case the condition $\mathcal{K}_5 = 0$ implies $c = -21h^2, d = 8h^2$ and $f = 27h^2$. As regard, in this case, the third condition we have Coefficient$[\mathcal{K}_6, x^6y^5] = -493000(b+54h^3)/9 = 0$, Coefficient$[\mathcal{K}_6, y^{11}] = 40(409a - 32b + 12178h^3)/9 = 0$ which gives $b = -54h^3$ and $a = -34h^3$ and this implies $\mathcal{K}_6 = 0$. Therefore we arrive at the family of systems $\dot{x} = -(2h + x)(17h^2 + 2hx - x^2 - 4hy + xy + y^2)$, $\dot{y} = -(3h - y)^2(6h + y)$ which after the translation of the origin of coordinates to the singular paint $(-2h, 3h)$ becomes

$$\dot{x} = x(-9hx + x^2 - xy - y^2), \quad \dot{y} = -y^2(9h + y). \tag{3.7}$$

Moreover we may assume $h \in \{0,1\}$ due to the rescaling $(x,y,t) \mapsto (hx, hy, t/h^2)$ in the case $h \neq 0$. For these systems we have $H(a,X,Y,Z) = -X^2(X-Y)Y^2(X-Y-9hZ)(Y+9hZ)$. Thus systems (3.7) possess 5 real invariant affine straight lines (two of them being double) of total multiplicity 7, and namely: $L_{1,2} = x$, $L_{3,4} = y$, $L_5 = y+9h$, $L_6 = x-y$, $L_7 = x-y-9h$. On the other hand systems (3.7) possess finite singularities of total multiplicity 9: $M_{1,2,3,4}(0,0)$ $M_{5,6}(9h,0)$, $M_7(0,-9h)$, $M_8(-9h,-9h)$, $M_9(9h,-9h)$.

Since $h \in \{0,1\}$ we get two configurations. More exactly, in the case $h=1$ we obtain *Config. 8.18* and in the case $h=0$ systems (3.7) become cubic homogeneous, possessing one triple and two double $\mathbb{IL}s$. This leads to *Config. 8.19* (see Figure 3.1).

B) Systems with cubic homogeneous parts (3.4)**.** In this case we may assume (due to a translation) $g = m = 0$ and therefore we consider the family of systems

$$\dot{x} = a + cx + dy + 2hxy + ky^2 - x^2y, \quad \dot{y} = b + ex + fy + lx^2 + ny^2 - x^2y \tag{3.8}$$

for which $C_3(x,y) = x^2(x-y)y$. For these systems we also could apply Remark 3.1 and we calculate $\mathcal{K}_4 = 2(y-x)\big((h-n)x^2 + 2kxy - ky^2\big)/9$. So the condition $\mathcal{K}_4 = 0$ yields $k=0$ and $n=h$ and then we have Coefficient$[\mathcal{K}_5, xy^3] = -40(4d+3h^2)/3 = 0$, Coefficient$[\mathcal{K}_5, x^2y^2] = 20(4c-3d-f-3h^2) = 0$. These relations imply $d = -3h^2/4$, $f = (16c-3h^2)/4$ and then we calculate $\mathcal{K}_5 = 10(39c+13e+12lh-12h^2)x^4/3 - 40(3c+e+3lh-3h^2)x^3y/3$. Hence the condition $\mathcal{K}_5 = 0$ gives $e = -3(c+hl-h^2)$ and $h(h-l) = 0$ and we consider two cases: $h=0$ and $l = h \neq 0$.

1) *The case* $h=0$. Then $\mathcal{K}_4 = \mathcal{K}_5 = 0$ and for systems (3.8) we calculate

$$\text{Coefficient}[\mathcal{K}_6, x^5y^6] = 6320a/9 = 0, \ \text{Coefficient}[\mathcal{K}_6, x^6y^5] = -40(881a+67b)/9 = 0,$$
$$\text{Coefficient}[\mathcal{K}_6, x^7y^4] = 8(10175a + 1675b + 7872c\,l)/9 = 0,$$

which gives $a = b = c\,l = 0$ and this implies $\mathcal{K}_6 = 0$. On the other hand we note that $c \neq 0$, otherwise systems (3.8) become degenerate. So $l=0$ and we get the family of systems

$$\dot{x} = cx - x^2y, \quad \dot{y} = -3cx + 4cy - x^2y, \tag{3.9}$$

where $c \neq 0$. Hence we may assume $c \in \{-1,1\}$ due to the rescaling $(x,y,t) \mapsto (\sqrt{|c|}x, \sqrt{|c|}y, t/c)$. For the above systems we calculate $H(X,Y,Z) = -4cX(X-Y)Z^2$ and by Lemma 2.4 these systems have invariant lines of total multiplicity 5 (here the line at infinity is a triple one). Since $c \in \{-1,1\}$ we deduce that in the case $h=0$ we could not obtain a configuration of invariant lines of type $(3,2,1,1)$.

2) *The case* $l = h \neq 0$. Then we have again $\mathcal{K}_4 = \mathcal{K}_5 = 0$ and for systems (3.8) we calculate $\mathcal{K}_6 = 10x^5(x-y)^5\big[(364a + 268b + 852ch + 67h^3)x - 8(79a + 791ch)y\big]/9$. So the condition $\mathcal{K}_6 = 0$ implies $a = -791ch/79 \equiv a_0$ and $b = h(220616c - 5293h^2)/21172 \equiv b_0$ and we arrive at the 2-parameter family of systems

$$\begin{aligned}\dot{x} =& a_0 + cx - 3h^2y/4 + 2hxy - x^2y,\\ \dot{y} =& b_0 - 3cx + hx^2 + (16c - 3h^2)y/4 - x^2y + hy^2,\end{aligned} \tag{3.10}$$

for which we have $H(X,Y,Z) = 2^{-6}67^{-2}79^{-3}Z\big[268h(X-Y)^2 - 1072c(X-Y)Z + h(5476c - 67h^2)Z^2\big]$. So the above systems have invariant lines of total multiplicity 4 and neither in the direction $x = 0$ nor in the direction $y = 0$. However in order to have the configuration $(3,2,1,1)$ we need at least one line in the direction $y = 0$. Considering the above systems and the equations (2.17) for the direction $y = 0$ we obtain $Eq_8 = -3c = 0$, i.e. $c = 0$. This leads to the systems $\dot{x} = (2x - h)(3h - 2x)y/4$, $\dot{y} = (y - h)(h^2 - 4x^2 + 4hy)/4$ for which $h \neq 0$ (otherwise we get a degenerate system). Then via the transformation $(x,y,t) \mapsto (h(2x+1)/2, h(y+1), t/h^2)$ we obtain the system

$$\dot{x} = (1-x)x(1+y), \quad \dot{y} = y(1 - x - x^2 + y), \tag{3.11}$$

for which we have $H(X,Y,Z) = X^2YZ(X-Y)(X-Z)(Y-X+Z)$. Thus system (3.11) besides the double line at infinity (see Lemma 2.4) possesses 5 affine real invariant straight lines of total multiplicity 6, and namely: $L_{1,2} = x$, $L_3 = y$, $L_4 = x - y$, $L_5 = x - y - 1$, $L_7 = x - 1$.

On the other hand we observe that systems (3.11) possess finite singularities of total multiplicity 6: $M_1(0,0)$, $M_{2,3}(0,-1)$, $M_4(1,0)$, $M_5(1,1)$, $M_6(-1,-1)$. Taking into account Lemma 2.1 for these systems we calculate: $\mu_0 = \mu_1 = \mu_2 = 0$, $\mu_3 = -x^2y$. Therefore since $\mu_3 \neq 0$ by Lemma 2.1 two finite singular points "have gone" to infinity and collapsed with the singular point $[0,1,0]$ located on the "end" of the invariant line $x = 0$ and one infinite singularity "has gone" to infinity and collapsed with the singular point $[1,0,0]$ located on the "end" of the invariant line $y = 0$. So we get the configuration given by *Config. 8.20.*

C) Systems with cubic homogeneous parts (3.5)**.** Due to a translation we may assume that in the quadratic parts of cubic systems the condition $g = m = 0$ holds. So we consider the family of systems

$$\dot{x} = a + cx + dy + 2hxy + ky^2 + x^3, \ \dot{y} = \ b + ex + fy + lx^2 + ny^2 + xy^2 \tag{3.12}$$

for which $C_3(x,y) = x^2(x-y)y$. Taking into consideration Remark 3.1 we impose the conditions $\mathcal{K}_4 = \mathcal{K}_5 = \mathcal{K}_6 = 0$ to be satisfied for systems (3.12). We calculate $\mathcal{K}_4 = 2x(lx^2 - 3hxy + hy^2 - 3ky^2)/9$ and therefore the condition $\mathcal{K}_4 = 0$ gives $h = k = l = 0$. Then we obtain Coefficient$[\mathcal{K}_5, xy^3] = -40d/3 = 0$, i.e. $d = 0$ and we get $\mathcal{K}_5 = 10x^2\big[9(3c - 4e - 3f)x^2 + 2(9c - 8e + 9n^2)xy + 6(c - 4f + 3n^2)y^2\big]/3$. Therefore the condition $\mathcal{K}_5 = 0$ implies $c = -3n^2$, $e = -9n^2/4$ and $f = 0$. Forcing the condition $\mathcal{K}_6 = 0$ we have Coefficient$[\mathcal{K}_6, x^5y^6] = -2680(a + 2n^3)/9 = 0$ and Coefficient$[\mathcal{K}_6, x^9y^2] = -1620(4b + 9n^3) = 0$. These relations give us $a = -2n^3$, $b = -9n^3/4$ which implies $\mathcal{K}_6 = 0$. However in this case we get a family of degenerate systems and hence, we conclude that systems (3.12) could not have invariant straight lines of total multiplicity eight.

Systems with potential configuration $(2, 2, 2, 1)$. As a first step we construct the cubic homogeneous parts of systems (3.1) for which the condition above is fulfilled. So we shall consider the family of systems (3.2) and we force the condition $\mathcal{V}_3 = 0$ to be satisfied. A straightforward computation of the value of $\mathcal{V}_3$ for systems (3.2) yields: $\mathcal{V}_3 = 32\sum_{j=0}^{4} \mathcal{V}_{3j} x^{4-j} y^j$, where

$$\begin{aligned} &\mathcal{V}_{30} = -u(3+u),\ \mathcal{V}_{31} = 2u(2-v),\ \mathcal{V}_{32} = 2 + 3r - 2ru + v - v^2,\\ &\mathcal{V}_{33} = -2r(1+v), \quad \mathcal{V}_{34} = -r^2. \end{aligned} \tag{3.13}$$

So the condition $\mathcal{V}_{34} =$ implies $r = 0$ and then $\mathcal{V}_{33} = 0$ and $\mathcal{V}_{32} = (2-v)(1+v)$. Taking into account (3.13) we consider two cases: $u \neq 0$ and $u = 0$.

1) *The case* $u \neq 0$. By (3.13), the conditions $\mathcal{V}_{3i} = 0$, $i = 0, 1, 2$ yield $u = -3$ and $v = 2$ and hence, we arrive at the cubic homogeneous system $\dot{x} = -2x^3 + x^2y$, $\dot{y} = -3x^2y + 2xy^2$.

2) *The case* $u = 0$. Considering (3.13) and the condition $r = 0$ we get $\mathcal{V}_{30} = \mathcal{V}_{31} = \mathcal{V}_{33} = \mathcal{V}_{33} = 0$ and $\mathcal{V}_{32} = (2-v)(1+v)$. In the case $v = 2$ we obtain the system

$$\dot{x} = x^3 + x^2y, \quad \dot{y} = 2xy^2, \tag{3.14}$$

which can be brought to the above system via the change $(x, y, t) \mapsto (x, x - y, -t)$. The case $v = -1$ leads to the system

$$\dot{x} = x^3 - 2x^2y, \quad \dot{y} = -xy^2. \tag{3.15}$$

Thus for the further investigation it remains to use two different homogeneous systems: (3.14) and (3.15). We observe that for system (3.14) we have $\mathcal{L}_7 = -8x^4 \neq 0$, whereas for system (3.15) the condition $\mathcal{L}_7 = 0$ holds. So we arrive at the following remark.

Remark 3.3. *If* $\mathcal{V}_3 = 0$ *then systems* (3.2) *due to a linear transformation can be brought to system* (3.14) *if* $\mathcal{L}_7 \neq 0$ *and to system* (3.15) *if* $\mathcal{L}_7 = 0$.

In what follows we construct the canonical systems which have either the cubic homogeneities (3.14) or (3.15).

A) Systems with cubic homogeneous parts (3.14)***.*** Due to a translation we may assume that in the quadratic part of cubic systems (3.1) the condition $g = n = 0$ holds. So we consider the family of systems

$$\dot{x} = a + cx + dy + 2hxy + ky^2 + x^3 + x^2y,\ \dot{y} = b + ex + fy + lx^2 + 2mxy + 2xy^2 \tag{3.16}$$

and by Remark 3.1 we impose the conditions $\mathcal{K}_4 = \mathcal{K}_2 = \mathcal{K}_8 = 0$ to be satisfied for these systems. We calculate $\mathcal{K}_4 = 2\big[lx^3 - (6h+m)x^2y + 2(2h-3k)xy^2 + 2ky^3\big]/9$ and hence the condition $\mathcal{K}_4 = 0$ yields $h = k = l = m = 0$. Then we have $\mathcal{K}_2 = -3x^4y^2(2x+y)^2(bx-ay)$ and $\mathcal{K}_8 = -10(4c-2e-f)x^4 - 6(c-d-10f)x^3y - 8(5c-5d-2f)x^2y^2$. Obviously the condition $\mathcal{K}_2 = 0$ implies $b = a = 0$, whereas the condition $\mathcal{K}_8 = 0$ gives $d = c$, $e = 2c$ and $f = 0$. So we get the systems

$$\dot{x} = (c+x^2)(x+y), \quad \dot{y} = 2x(c+y^2), \tag{3.17}$$

for which we calculate: $H(X,Y,Z) = -\,2(X-Y)(X^2+cZ^2)^2(Y^2+cZ^2)$. We observe that $c \neq 0$, otherwise we get a degenerate system. Moreover due to the rescaling $(x,y,t) \mapsto (\sqrt{|c|}x, \sqrt{|c|}y, t/c)$ we may consider $c \in \{-1,1\}$. The above systems possess invariant affine straight lines of total multiplicity 7, and namely: $x^2+c=0$ (both double and either real if $c=-1$, or complex if $c=1$), $y^2+c=0$ (both simple and either real if $c=-1$ or complex if $c=1$) and $y=x$ (simple).

Considering Lemma 2.1 for these systems we calculate: $\mu_0 = \mu_1 = 0$, $\mu_2 = -8cx^2$. If $c \neq 0$ by the same lemma two finite singular points have gone to infinity and collapsed with the singular point $[0,1,0]$ located on the "end" of the invariant line $x=0$. Moreover systems (3.17) became degenerate if $c=0$. As a result we obtain *Config. 8.21* in the case $c=-1$ and *Config. 8.22* in the case $c=1$ (see Figure 3.1).

B) Systems with cubic homogeneous parts (3.15)**.** In this case due to a translation we may assume for systems (3.1) the condition $g = n = 0$ holds. So we consider the family of systems

$$\dot{x} = a+cx+dy+2hxy+ky^2+x^3-2x^2y, \quad \dot{y} = b+ex+fy+lx^2+2mxy-xy^2, \tag{3.18}$$

for which considering Remark 3.1 we calculate $\mathcal{K}_4 = -2\big[2clx^3 + (4m-3h)x^2y - (h+3k)xy^2 + ky^3\big]/9$. So the condition $\mathcal{K}_4 = 0$ yields $h = k = l = m = 0$ and then we have $\mathcal{K}_2 = -3x^4(x-y)^2y^2(bx-ay)$, $\mathcal{K}_8 = 5(4c-f)x^4 - (16c+3d-40f)x^3y + 4(4c+7d-10f)x^2y^22$. Therefore the condition $\mathcal{K}_2 = 0$ implies $b = a = 0$ and the condition $\mathcal{K}_8 = 0$ gives $c = d = f = 0$ and this leads to degenerate systems. Thus we conclude that cubic systems (3.18) could not possess invariant straight lines in the potential configuration of type (2, 2, 2, 1).

3.1.2. Invariant criteria for the realization of the configurations with three distinct infinite singularities

I. Conditions for *Config. 8.18, 8.19* and *8.20*. It was shown earlier that systems (3.1) could possess the potential configuration of invariant straight lines $(3,2,1,1)$ only if their homogeneous cubic parts via a linear transformation could be brought to the form (3.3) (respectively (3.4)) and for this the conditions $\mathcal{V}_4 = \mathcal{V}_5 = 0$ and $\mathcal{K}_7 \neq 0$ (respectively $\mathcal{K}_7 = \mathcal{V}_6 = 0$) necessarily must hold (see Remark 3.2).

1) The case $\mathcal{K}_7 \neq 0$. Then for $\mathcal{V}_4 = \mathcal{V}_5 = 0$ and $\mathcal{K}_7 \neq 0$ a cubic system (3.1) due to an affine transformation could be brought to the form (3.6). And forcing the coefficients of these systems to satisfy the conditions $\mathcal{K}_4 = \mathcal{K}_5 = \mathcal{K}_6 = 0$ (see Remark 3.1) and using an additional translation we arrive at the systems (3.7). These systems possess two configurations, given by the values of the parameter h. So to distinguish *Config. 8.18* ($h \neq 0$) from configuration *Config. 8.19* ($h = 0$) it remains to determine the invariant polynomial which is responsible for this condition. We calculate $\mathcal{L}_1 = 41472hy^2(y - x)$ and therefore the condition $\mathcal{L}_1 = 0$ is equivalent to $h = 0$. Thus if for systems (3.6) the conditions $\mathcal{K}_4 = \mathcal{K}_5 = \mathcal{K}_6 = 0$ are satisfied then we get the configuration given by *Config. 8.18* if $\mathcal{L}_1 \neq 0$ and by *Config. 8.19* if $\mathcal{L}_1 = 0$.

2) The case $\mathcal{K}_7 = 0$. As it was mentioned above in this case the condition $\mathcal{V}_6 = 0$ must hold and we consider systems (3.12).

Forcing the coefficients of these systems to satisfy the condition $\mathcal{K}_4 = \mathcal{K}_5 = 0$ we get for the coefficients of quadratic terms the relations: $k = 0$, $n = h$ and $h(h - l) = 0$. It was proved (see page 86) that in the case $h = 0$ systems (3.8) could not possess a configuration of type $(3, 2, 1, 1)$. So we have to distinguish the case $h \neq 0$ from $h = 0$. On the other hand for systems (3.8) with $k = 0$ and $n = h$ we have $\mathcal{L}_1 = 4608hx(x - y)^2$. and Hence this invariant polynomial governs the condition $h \neq 0$.

If $\mathcal{L}_1 \neq 0$ then due to the additional condition $\mathcal{K}_6 = 0$ systems (3.8) become as systems (3.10) for which the condition $c = 0$ has to be satisfied. On the other hand for these systems we calculate $\mathcal{L}_6 = -16c$ and hence $\mathcal{L}_6 = 0$ is equivalent to $c = 0$.

Thus systems (3.1) possess the (unique) configuration *Config. 8.20* if and only if the conditions $\mathcal{V}_4 = \mathcal{V}_5 = \mathcal{V}_6 = 0$, $\mathcal{K}_4 = \mathcal{K}_5 = \mathcal{K}_6 = \mathcal{K}_7 = \mathcal{L}_6 = 0$ and $\mathcal{L}_1 \neq 0$ are satisfied. This complete the proof of the statement B_1) of Main Theorem B.

II. Conditions for *Config. 8.21* and *8.22*. By Remark 3.3 the condition $\mathcal{L}_7 \neq 0$ must be satisfied for systems (3.2) and therefore, we arrive at systems (3.16) (after a translation). So forcing the coefficients of these these systems to satisfy the condition $\mathcal{K}_4 = \mathcal{K}_2 = \mathcal{K}_8 = 0$ we arrive at the systems (3.17) with $c \in \{-1, 1\}$. For these systems we calculate $K_9 = -180cx^4y^2$ and as $c \neq 0$ we conclude that $\text{sign}\,(\mathcal{K}_9) = -\text{sign}\,(c)$.

Thus systems (3.1) belong to $\mathbb{CSL}_8$ if and only if the conditions $\mathcal{V}_3 = \mathcal{K}_4 = \mathcal{K}_2 = \mathcal{K}_8 = 0$ and $\mathcal{L}_7 \neq 0$ are satisfied. Moreover we have the configuration given by *Config. 8.21* if $K_9 > 0$ and by *Config. 8.22* if $K_9 < 0$ (see Figure 3.1).

This complete the proof of the statement B_2) of Main Theorem B.

3.1.3. Perturbations of normal forms

To finish the proof of the Main Theorem B it remains to construct for the normal forms given in this theorem the corresponding perturbations, which prove that the respective invariant straight lines have the indicated multiplicities. Here we construct such perturbations and for each configuration *Configs. 8.j*, $j = 18, 19, 20, 21, 22$ we give: (i) the corresponding normal form and its invariant straight lines; (ii) the respective perturbed normal form with its invariant straight lines and (iii) the configuration *Configs. 8.j*$_\varepsilon$, $j = 18, 19, 20, 21, 22$ corresponding to the perturbed system.

Config. 8.18: $$\begin{cases} \dot{x} = -x(9hx - x^2 + xy + y^2), \\ \dot{y} = -y^2(9h + y); \end{cases}$$

Invariant lines: $L_{1,2} = x,\ L_{3,4} = y,\ L_5 = y + 9,\ L_6 = x - y,\ L_7 = x - y - 9;$

Config. 8.18$_\varepsilon$: $$\begin{cases} \dot{x} = -\dfrac{81\varepsilon}{(1-\varepsilon)^2}x - 9x^2 - 9\varepsilon xy + x^3 + (\varepsilon - 1)x^2y - (1 + 2\varepsilon)xy^2, \\ \dot{y} = -\dfrac{1+\varepsilon}{(1-\varepsilon)^2}y\big[y(\varepsilon - 1)y - 9\big]\big[y(\varepsilon - 1)y + 9\varepsilon\big]; \end{cases}$$

Invariant lines: $$\begin{cases} L_1 = x,\ L_2 = (1-\varepsilon)(x + \varepsilon y) + 9\varepsilon,\ L_3 = y,\ L_4 = (1-\varepsilon)y - 9\varepsilon, \\ L_5 = (1-\varepsilon)y + 9,\ L_6 = (1-\varepsilon)(x - y) + 9\varepsilon,\ L_7 = (1-\varepsilon)(x-y) - 9. \end{cases}$$

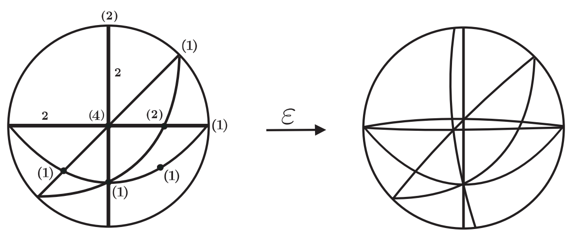

Fig. 3.3. Perturbation of normal form corresponding to the configuration *Config. 8.18*

Config. 8.19: $$\begin{cases} \dot{x} = x(x^2 - xy - y^2), \\ \dot{y} = -y^3; \end{cases}$$

Invariant lines: $L_{1,2} = x,\ L_{3,4,5} = y,\ L_{6,7} = x - y;$

Config. 8.19$_\varepsilon$: $$\begin{cases} \dot{x} = -\dfrac{81\varepsilon^3}{(1-\varepsilon)^2}x - 9\varepsilon x^2 - 9\varepsilon^2 xy + x^3 + (\varepsilon - 1)x^2y - (1 + 2\varepsilon)xy^2, \\ \dot{y} = -\dfrac{1+\varepsilon}{(1-\varepsilon)^2}y\big[y(\varepsilon - 1)y - 9\varepsilon\big]\big[y(\varepsilon - 1)y + 9\varepsilon^2\big]; \end{cases}$$

Invariant lines: $\begin{cases} L_1 = x,\ L_2 = (1-\varepsilon)(x+\varepsilon y) + 9\varepsilon^2,\ L_3 = y,\ L_4 = (1-\varepsilon)y + 9\varepsilon, \\ L_5 = (1-\varepsilon)y - 9\varepsilon^2,\ L_6 = (1-\varepsilon)(x-y)9\varepsilon,\ L_7 = (1-\varepsilon)(x-y) + 9\varepsilon^2; \end{cases}$

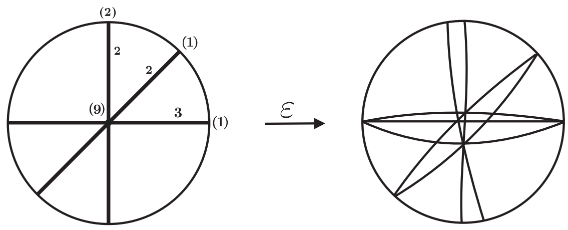

Fig. 3.4. Perturbation of normal form corresponding to the configuration *Config. 8.19*

Config. 8.20: $\begin{cases} \dot{x} = (1-x)x(1+y), \\ \dot{y} = y(1-x+y-x^2); \end{cases}$

Invariant lines: $L_{1,2} = x,\ L_3 = y,\ L_4 = x-y,\ L_5 = x-y-1,\ L_6 = x-1,\ L_7:\ Z = 0;$

Config. 8.20$_\varepsilon$: $\begin{cases} \dot{x} = x\Big[\dfrac{1+\varepsilon}{(1+2\varepsilon)^2} - x + y + \varepsilon x^2 - (1+2\varepsilon)xy\Big], \\ \dot{y} = y\Big[\dfrac{1+\varepsilon}{(1+2\varepsilon)^2} - x + y + (\varepsilon-1)x^2 - 3\varepsilon xy + \varepsilon y^2\Big]; \end{cases}$

Invariant lines: $\begin{cases} L_1 = x,\ L_2 = x+\varepsilon y,\ L_3 = y,\ L_4 = x-y,\ L_5 = (1+2\varepsilon)(x-y) - 1, \\ L_6 = (1+2\varepsilon)x - 1,\ L_7 = \varepsilon(1+2\varepsilon)(x-y) - 1 - \varepsilon. \end{cases}$

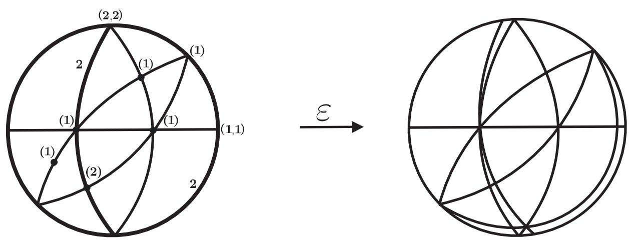

Fig. 3.5. Perturbation of normal form corresponding to the configuration *Config. 8.20*

Config. 8.21: $\begin{cases} \dot{x} = (x^2-1)(x+y), \\ \dot{y} = 2x(y^2-1); \end{cases}$

Invariant lines: $L_{1,2} = x-1,\ L_{3,4} = x+1,\ L_5 = y-1,\ L_6 = y+1,\ L_7 = x-y;$

$$\text{Config. } 8.21_\varepsilon: \quad \begin{cases} \dot{x} = \dfrac{1}{1+\varepsilon}\big[(1+\varepsilon)x^2-1)\big]\big[x+(1+4\varepsilon)y\big], \\ \dot{y} = \dfrac{2}{1+\varepsilon}\big[(1+\varepsilon)x+\varepsilon y\big]\big[(1+\varepsilon)y^2-1)\big]; \end{cases}$$

$$\text{Invariant lines: } \begin{cases} L_{1,3} = (1+\varepsilon)x^2-1,\ L_{2,4} = (1+\varepsilon)(x+2\varepsilon y)^2-(1+2\varepsilon)^2, \\ L_{5,6} = (1+\varepsilon)y^2-1,\ L_7 = x-y. \end{cases}$$

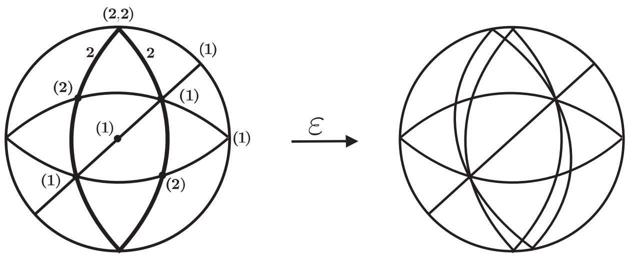

Fig. 3.6. Perturbation of normal form corresponding to the configuration *Config. 8.21*

$$\text{Config. } 8.22: \quad \begin{cases} \dot{x} = (x^2+1)(x+y), \\ \dot{y} = 2x(y^2+1); \end{cases}$$

Invariant lines: $L_{1,2} = x-i,\ L_{3,4} = x+i,\ L_5 = y-i,\ L_6 = y+i,\ L_7 = x-y;$

$$\text{Config. } 8.22_\varepsilon: \quad \begin{cases} \dot{x} = \dfrac{1}{1+\varepsilon}\big[(1+\varepsilon)x^2+1)\big]\big[x+(1+4\varepsilon)y\big], \\ \dot{y} = \dfrac{2}{1+\varepsilon}\big[(1+\varepsilon)x+\varepsilon y\big]\big[(1+\varepsilon)y^2+1)\big]; \end{cases}$$

$$\text{Invariant lines: } \begin{cases} L_{1,3} = (1+\varepsilon)x^2+1,\ L_{2,4} = (1+\varepsilon)(x+2\varepsilon y)^2+(1+2\varepsilon)^2, \\ L_{5,6} = (1+\varepsilon)y^2+1,\ L_7 = x-y. \end{cases}$$

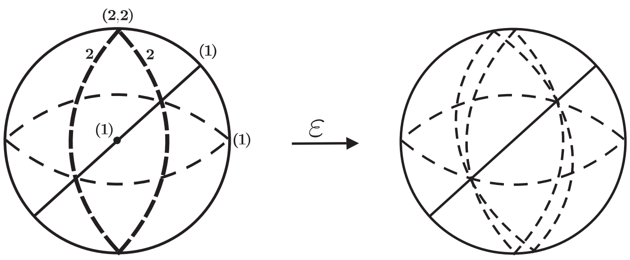

Fig. 3.7. Perturbation of normal form corresponding to the configuration *Config. 8.22*

3.2. Cubic systems with exactly one infinite singularity

Assume that a cubic system possesses exactly one infinite singularity which is determined by one real factor of degree four of the polynomial C_3. Then considering Lemma 2.2 we obtain that systems (2.1) via a linear transformation become:

$$\begin{aligned} x' &= p_0 + p_1(x,y) + p_2(x,y) + ux^3 + vx^2y + rxy^2, \\ y' &= q_0 + q_1(x,y) + q_2(x,y) - x^3 + ux^2y + vxy^2 + ry^3 \end{aligned} \tag{3.19}$$

with $C_3 = x^4$. Hence, the infinite singular point is located at the "end" of the straight lines $x = 0$.

3.2.1. Construction of cubic homogeneities

We split our examination depending on the possible potential types of configurations, i.e. we consider systems (3.19) and apply Theorem 2.2.

1) Configuration (3, 3, 1). By Theorem 2.2 in order to have the configuration (3, 3, 1) the necessary condition $\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{U}_1 = 0$ must be fulfilled. So we calculate

$$\mathcal{V}_1 = 32(ux^2 + vxy + ry^2)^2, \ \mathcal{V}_2 = 96rx^2, \ \mathcal{U}_1 = 128(v^2 - 4ru).$$

Evidently that the condition $\mathcal{V}_1 = 0$ which is equivalent to $u = v = r = 0$ implies $\mathcal{V}_2 = \mathcal{U}_1 = 0$. Therefore we get the following form of cubic systems

$$x' = p_0 + p_1 + p_2, \quad y' = q_0 + q_1 + q_2 - x^3. \tag{3.20}$$

2) Configuration (3, 2, 1, 1). By the same theorem in this case the conditions $\mathcal{V}_4 = \mathcal{V}_5 = \mathcal{U}_2 = 0$ necessarily hold. Considering (3.19) we force these conditions to be satisfied. We obtain Coefficient$[\mathcal{V}_5, 9x^4/32] = -3r^2 + 2ruv - v^3 = 0$, Coefficient$[\mathcal{V}_5, 9xy^3/128] = -r^3 = 0$. The above conditions imply $r = v = 0$ which leads to $\mathcal{V}_5 = 0$ and this gives $\mathcal{U}_2 = \mathcal{V}_4 = 0$. Thus we have systems with the following cubic part

$$x' = p_0 + p_1 + p_2 + ux^3, \quad y' = q_0 + q_1 + q_2 - x^3 + ux^2y \tag{3.21}$$

for which we calculate $\mathcal{V}_1 = 32u^2x^4$. We detect that if $u = 0$, i.e. $\mathcal{V}_1 = 0$, we get systems (3.20). If $u \neq 0$, i.e. $\mathcal{V}_1 \neq 0$, applying the rescaling $(x, y, t) \mapsto (ux, y, t/u^3)$ we can set $u = 1$ and we get the systems

$$x' = p_0 + p_1 + p_2 + x^3, \quad y' = q_0 + q_1 + q_2 - x^3 + x^2y. \tag{3.22}$$

3) Configuration (2, 2, 2, 1). By Theorem 2.2 in order to have the mentioned configuration the necessary condition $\mathcal{V}_3 = 0$ must be satisfied. We get Coefficient$[\mathcal{V}_3, y^4/32] = -r^2$, Coefficient$[\mathcal{V}_3, x^2y^2/32] = -2ru - v^2$, Coefficient$[\mathcal{V}_3, x^4/32] = 3v - u^2$. Thus $\mathcal{V}_3 = 0$ is equivalent to $u = v = r = 0$ and therefore we obtain again systems (3.20).

Lemma 3.1. *Assume that for a homogeneous cubic system the conditions* $\mathcal{D}_1 = \mathcal{D}_2 = \mathcal{D}_3 = 0$ *hold. Then via a linear transformation and time rescaling this system can be brought to the one of the canonical form* $(\tilde{P}, \tilde{Q})$ *indicated bellow if and only if the corresponding conditions are satisfied. Moreover the cubic systems with these homogeneities can have only the configurations of invariant straight lines of the type given next to each homogeneity:*

(i) $\mathcal{V}_1 = 0 \;\Rightarrow\; (\tilde{P}, \tilde{Q}) = (0, -x^3) \;\Rightarrow\; (3,3,1),\ (3,2,1,1),\ (2,2,2,1)$*;*

(ii) $\mathcal{V}_1 \neq 0,\ \mathcal{V}_5 = 0 \;\Rightarrow\; (\tilde{P}, \tilde{Q}) = (x^3, -x^3 + x^2y) \;\Rightarrow\; (3,2,1,1)$.

Also we need the following result:

Remark 3.4. *Any invariant line of the form* $x + \alpha = 0$ *(i.e. in the direction* $x = 0$*) of cubic systems* (2.1) *must be a factor of the polynomials* $P(x, y)$*, i.e.* $(x + \alpha) \mid P(x, y)$.

Indeed, according to the definition, for an invariant line $ux + vy + w = 0$ we have $uP + vQ = (ux + vy + w)R(x, y)$, where the cofactor $R(x, y)$ generically is a polynomial of degree two. In our particular case (i.e. $u = 1, v = 0, w = \alpha$) we obtain $P(x) = (x + \alpha)R(x)$, which means that $(x + \alpha)$ divides $P(x)$. We mention that this remark could be applied for any cubic system when we examine the direction $x = 0$. Similarly, for an invariant line $y + \beta = 0$ in the the direction $y = 0$ it is necessary $(y + \beta) \mid Q(x, y)$.

3.2.2. Construction of canonical forms and of the corresponding configurations of invariant lines

In what follows we consider two families of systems, the cubic homogeneities of which are given by Lemma 3.1. Since each one of these cubic homogeneities leads to the given type of configurations (see Lemma 3.1), in order to construct the corresponding canonical form we shall apply first the respective necessary conditions given by Remark 3.1.

3.2.2.1. Construction of the normal form with cubic homogeneities $(0, -x^3)$

In this case due to a translation of the origin of coordinates we can consider $l = 0$ and hence we get the cubic systems

$$\dot{x} = a + cx + dy + gx^2 + 2hxy + ky^2, \quad \dot{y} = b + ex + fy + 2mxy + ny^2 - x^3 \tag{3.23}$$

for which we have $H(X, Y, Z) = Z$. So we force the necessary conditions given by Remark 3.1 which correspond to each type of configuration. We divide our examination in three subcases defined by (a_1)–(a_3) of the mentioned remark.

(a_1) For systems (3.23) we calculate: $\mathcal{L}_1 = \mathcal{K}_1 = 0$ and

$$\mathcal{L}_2 = -20736[(4h^2 + 14km + hn + 4n^2)x^2 - 7k(h - n)xy - 7k^2y^2) = 0.$$

The above condition implies $k = 0$ and as the discriminant of the binary form $4h^2 + hn + 4n^2$ is negative we obtain $h = n = 0$.

(a_2) In the same manner in the case of the configuration $(3, 2, 1, 1)$ we determine $\mathcal{K}_4 = \mathcal{K}_6 = 0$ and $\mathcal{K}_5 = -120kx^3(2hx + nx + ky)$. Therefore $\mathcal{K}_5 = 0$ is equivalent to $k = 0$.

(a_3) We calculate $\mathcal{K}_4 = 0$, $\mathcal{K}_2 = -2x^4(nx - ky)(mx^2 - hxy + nxy - ky^2)^2$ and $\mathcal{K}_8 = 12k(h + n)x^4$. We detect that the condition $\mathcal{K}_2 = 0$ is equivalent to either $k = n = 0$ or $k = m = 0$ and $n = -h$ and this implies $\mathcal{K}_8 = 0$.

From the above results the next proposition follows.

Proposition 3.1. *Assume that a cubic systems* (3.23) *possesses a potential configuration of a given type. Then for this system the following conditions hold, respectively:*

$$\begin{array}{lll} (3,\,3,\,1) \Rightarrow k = n = h = 0 & \Leftrightarrow & \mathcal{L}_2 = 0; \\ (3,\,2,\,1,\,1) \Rightarrow k = 0 & \Leftrightarrow & \mathcal{K}_5 = 0; \\ (2,\,2,\,2,\,1) \Rightarrow k = n = 0 \;\; \textit{or} \;\; k = m = h - h = 0 & \Leftrightarrow & \mathcal{K}_2 = 0. \end{array}$$

Systems with configuration (3,3,1). So considering systems (3.23) for $k = n = h = 0$ we calculate the equations (2.17) (setting $U = 1,\ V = 0$, see Remark 2.1):

$$Eq_1 = \ldots = Eq_8 = 0, \quad Eq_9 = d,\ Eq_{10} = a - cW + gW^2.$$

Remark 3.5. *According to [129] a cubic system could not have invariant line of multiplicity greater than or equal to 8.*

So by this remark we need to have at least one invariant affine line and therefore in what follows we assume $d = 0$. This leads to the family of systems

$$\dot{x} = a + cx + gx^2 \equiv P(x), \quad \dot{y} = b + ex + fy + 2mxy - x^3 \equiv Q(x, y) \tag{3.24}$$

for which we have $H(X, Z) = Z^2(gX^2 + cXZ + aZ^2)$. We conclude that the degree of H is four and we need to increase this degree up to seven, i.e. we have to find out a common factor of degree three of the polynomials $\mathcal{G}_i/H,\ i = 1, 2, 3$ (see Lemmas 2.3,2.4). We calculate

$$\mathcal{G}_1/H \equiv F_1(X, Y, Z), \quad \mathcal{G}_2/H = ZP^*(X, Z)F_2(X, Y, Z), \quad \mathcal{G}_3/H = 24Z^2[P^*(X, Z)]^3$$

where $P^*(X, Z)$ is the homogenized form of the polynomial P(x) and $F_1(X, Y, Z)$ (respectively $F_2(X, Y, Z)$) is a polynomial of degree 4 (respectively 8) with respect to the variables.

It is clear that systems (3.24) are degenerate if and only if the polynomials $P(x)$ and $Q(x, y)$ have a non-constant common factor (depending on x), i.e. the following condition must hold:

$$\Phi(y) \equiv R_x^{(0)}(P(x), Q(x, y)) \neq 0. \tag{3.25}$$

We remark that the required common factor of the polynomials $\mathcal{G}_i/H,\ i=1,2,3$ could contain as a factor $Z^\lambda,\ \lambda=0,\dots,3$. Next we split our examination depending on the value of the parameter λ.

1) The case $\lambda=3$. Then the following condition must hold:

$$R_Z^{(0)}(Z^3,F_1)=R_Z^{(1)}(Z^3,F_1)=R_Z^{(2)}(Z^3,F_1)=0.$$

For a system (3.24) we calculate $R_X^{(0)}(Z^3,F_1)=(g+2m)^3X^{12}$ which vanishes if and only if $g=-2m$. In this case we get

$$R_Z^{(1)}(Z^3,F_1)=X^4[(2c+f)X-8m^2Y]^2,\ R_Z^{(2)}(Z^3,F_1)=X[3aX-4m(eX+2fY)].$$

and the condition $R_Z^{(1)}(Z^3,F_1)=R_Z^{(2)}(Z^3,F_1)=0$ gives $m=a=f+2c=0$. Thus for $g=m=a=0$ and $f=-2c$ we get systems $\dot{x}=cx,\ \dot{y}=b+ex-x^3-2cy$ for which we have $\Phi=c^3(b-2cy)\neq 0$. Since $c\neq 0$ applying the transformation $(x,y,t)\mapsto \big(x,(b+2y)/(2c),t/c\big)$ we can set $c=1$ and $b=0$ which leads to the systems $\dot{x}=x,\ \ \dot{y}=ex-2y-x^3$ with $H(X,Z)=XZ^6$. Moreover we may assume $e=0$ due to the transformation $(x,y,t)\mapsto \big(x,y+ex/3,t\big)$ and so, the above systems became

$$\dot{x}=x,\quad \dot{y}=-2y-x^3 \tag{3.26}$$

On the other hand considering Lemma 2.1 for these systems we calculate: $\mu_0=\ldots=\mu_7=0,\ \mu_8=-2x^8$. By Lemma 2.1 eight finite singular points from 9 have gone to infinity and collapsed with the singular point $[0,1,0]$ located on the "end" of the invariant line $x=0$. Moreover the remaining finite singular point is $(0,0)$.

Thus a system (3.26) possess invariant straight lines: $L_1:\ x=0,\ \ L_{2,\dots 7}:Z=0$ and we get the configuration *Config. 8.48* (see Figure 3.2).

2) The case $\lambda=2$. Then the condition $R_Z^{(0)}(Z^2,F_1)=R_Z^{(1)}(Z^2,F_1)=0$ must be satisfied. We calculate

$$R_Z^{(0)}(Z^2,F_1)=(g+2m)^2X^8,\ R_Z^{(1)}(Z^2,F_1)=X^2[(2c+f)X+2m(g-2m)Y].$$

It is evident that $g+2m=2c+f=m=0$ (i.e. $g=m=0,f=-2c$) is equivalent to $R_Z^{(0)}(Z^2,F_1)=R_Z^{(1)}(Z^2,F_1)=0$. Moreover here we consider $a\neq 0$ otherwise we arrive at the previous case. So considering these conditions we get systems for which the polynomial H has degree 6 and therefore we need one additional linear common factor of the polynomials $\mathcal{G}_i/H, i=1,2,3$. For the corresponding systems we calculate

$$R_Z^{(0)}\big(F_1/Z^2,P^*(X,Z)\big)=-3Z[2c^4Y-(a^3+bc^3-ac^2e)Z]$$

which cannot vanishes (since $a\neq 0$). Thus, in the case $\lambda=2$ we cannot obtain systems with invariant lines of total multiplicity eight.

3) The case $\lambda = 1$. We have $F_1 \mid_{Z=0} = (g+2m)X^4$ and it is evident that the condition $g = -2m$ gives $F_1 = Z\tilde{F}_1$, where $\tilde{F}_1 = (2c+f)X^3 - 8m^2X^2Y + Zf(X,Y,Z)$, i.e. the additional condition $(2c+f)^2 + m^2 \neq 0$ must hold. In this case we obtain systems for which the polynomial H has degree 5 but must be 7. Therefore we need a common factor of degree 2 depending on X of the mentioned polynomials, i.e. the condition $R_X^{(0)}\big(\tilde{F}_1, [P^*(X,Z)]^3\big) = R_X^{(1)}\big(\tilde{F}_1, [P^*(X,Z)]^3\big) = 0$ must be satisfied. We calculate

$$\text{Coefficient}[R_X^{(0)}\big(\tilde{F}_1, [P^*(X,Z)]^3\big), Y^6Z^{12}] = 512m^9(cf+f^2-2am)^3(2c^2-cf-f^2+18am)^3 = 0.$$

This condition is equivalent to $m(cf+f^2-2am)(2c^2-cf-f^2+18am) = 0$ from which it results the following two subcases:

a) *The subcase* $m \neq 0$. Then $(cf+f^2-2am)(2c^2-cf-f^2+18am) = 0$ and we have two possibilities.

First we examine the possibility $cf+f^2-2am = 0$. Then $a = f(c+f)/(2m)$ and in this case we get

$$\text{Coefficient}[R_X^{(0)}\big(\tilde{F}_1, [P^*(X,Z)]^3\big), Y^3Z^{15}] = 8(c+2f)^9(f^3-4efm^2+8bm^3)^3 = 0,$$
$$\Phi = (f^3-4efm^2+8bm^3)\tilde{\varphi}(y) \neq 0.$$

We detect that the above conditions yield $c = -2f$ $\big($which gives $a = -f^2/(2m)\big)$ and this implies

$$R_X^{(0)}\big(\tilde{F}_1, [P^*(X,Z)]^3\big) = 0, R_X^{(1)}\big(\tilde{F}_1, [P^*(X,Z)]^3\big) = -243(f^3-4efm^2+8bm^3)^5Z^{10}/(16m^4) \neq 0.$$

So in this case we could not obtain systems with invariant lines of total multiplicity 8.

Now we consider the possibility $2c^2-cf-f^2+18am = 0$, i.e. $a = (f-c)(2c+f)/(18m)$. Then we obtain

$$R_X^{(0)}\big(\tilde{F}_1, [P^*(X,Z)]^3\big) = 0, \text{Coefficient}[R_X^{(1)}\big(\tilde{F}_1, [P^*(X,Z)]^3\big), Y^5Z^5] = -2^{18}3^{-2}(c+2f)^5m^{11} = 0.$$

Since $m \neq 0$ from the above equality it results $c = -2f$ which yields $a = -f^2/(2m)$ and therefore we arrive at the previous possibility.

b) *The subcase* $m = 0$. Then $2c+f \neq 0$ and we obtain

$$R_X^{(0)}\big(\tilde{F}_1, [P^*(X,Z)]^3\big) = -(c-f)^3(2c+f)^3Z^{15}[c^3fY + (a^3+bc^3-ac^2e)Z]^3 = 0,$$
$$\Phi = a^3+bc^3-ac^2e+c^3fy \neq 0.$$

Since $\Phi \neq 0$ and $2c+f \neq 0$ the above equality is equivalent to $f = c \neq 0$ and we calculate $R_X^{(1)}\big(\tilde{F}_1, [P^*(X,Z)]^3\big) = 27c^7(3a^2-c^2e)^2Z^{10} = 0$, i.e. $e = 3a^2/c^2$. So we arrive at the systems

$$\dot{x} = a + cx, \quad \dot{y} = b + cy + 3a^2x/c^2 - x^3$$

for which we have $R_X^{(0)}\big(\mathcal{G}_3/H, \mathcal{G}_1/H\big) = -16ac^3Z^5$. Therefore the condition $a \neq 0$ do not allow us to have invariant lines of total multiplicity 9, including the line at infinity. In this case we apply the transformation $(x, y, t) \mapsto (-a(3x+1)/c, (2a^3 - bc^3 - 27a^3y)/c^4, t/c)$ and the above systems became of the form

$$\dot{x} = x, \quad \dot{y} = y - x^2 - x^3 \tag{3.27}$$

with $H(X, Z) = X^3Z^4$.

Considering Lemma 2.1 for these systems we calculate: $\mu_0 = \ldots = \mu_7 = 0, \quad \mu_8 = x^8$. By the same lemma eight finite singular points from 9 have gone to infinity and collapsed with the singular point $[0, 1, 0]$ located on the "end" of the invariant line $x = 0$. On the other hand system (3.27) possesses the finite singular point $(0, 0)$ and the invariant straight lines: $L_{1,2,3}: \ x = 0, \quad L_{4,\ldots 7}: Z = 0$. This leads to *Config. 8.49* given in Figure 3.2.

4) The case $\lambda = 0$. Then Z does not divide F_1, i.e. the condition $g + 2m \neq 0$ must be satisfied and therefore we impose the condition $R_X^{(0)}(F_1, [P^*(X, Z)]^3) = R_X^{(1)}(F_1, [P^*(X, Z)]^3) = R_X^{(2)}(F_1, [P(X, Z)]^3) = 0$ to be satisfied. We calculate

$$\text{Coefficient}[R_X^{(2)}(F_1, [P^*(X, Z)]^3), Y^4Z^4] = 16g^6(g - 2m)^4m^4 = 0$$

which due to $g + 2m \neq 0$ implies three subcases: $g = 0, \ m \neq 0, \ m = 0, \ g \neq 0$ and $g = 2m \neq 0$.

a) If $g = 0$ (and $m \neq 0$) then we get $\text{Coefficient}[R_X^{(2)}(F_1, [P^*(X, Z)]^3), YZ^7] = -32c^6m^5 = 0$, i.e. $c = 0$ (due to $m \neq 0$). However in this case we obtain the contradictory conditions $R_X^{(0)}(F_1, [P^*(X, Z)]^3) = 64a^{12}m^6Z^{24} = 0$ $(m \neq 0)$ and $\Phi = a^3 \neq 0$.

b) If $m = 0$ (and $g \neq 0$) then

$$\text{Coefficient}[R_X^{(2)}(F_1, [P^*(X, Z)]^3), Y^2Z^6] = -f^2g^8[9c^2 + 12cf - 9f^2 - 4g(3a + 2eg)] = 0$$

and this condition implies other $f = 0$ or $9c^2 + 12cf - 9f^2 - 4g(3a + 2eg) = 0$.

So we suppose first $f = 0$ which implies

$$\text{Coefficient}[R_X^{(0)}(F_1, [P^*(X, Z)]^3), Z^{24}] = -g^3(c^2 - 4ag)^3\Phi^3 = 0.$$

Since $\Phi \neq 0$ the above condition leads to $a = c^2/(4g)$. In this case we get

$$R_X^{(1)}(F_1, [P^*(X, Z)]^3) = -(c^3 - 4ceg^2 + 8bg^3)^5Z^{15}/(1024g) = 0$$

which contradicts the condition $\Phi = (c^3 - 4ceg^2 + 8bg^3)^2/(64g^3) \neq 0$.

Now we examine the condition $9c^2 + 12cf - 9f^2 - 4g(3a + 2eg) = 0$, and $f \neq 0$, i.e. $a = (9c^2 + 12cf - 9f^2 - 8eg^2)/(12g)$. Then

$$\text{Coefficient}[R_X^{(0)}(F_1, [P^*(X, Z)]^3), Y^6Z^{18}] = 8f^6g^{12}(3c^2 + 6cf - 3f^2 - 4eg^2)^3/27 = 0$$

which is equivalent to $e = 3(c^2 + 2cf - f^2)/(4g^2)$. However in this case we get

$$\text{Coefficient}[R_X^{(1)}(F_1, [P^*(X, Z)]^3), Y^5Z^{10}] = -32f^5g^{14} \neq 0.$$

Thus, if $m = 0,\ g \neq 0$ then we cannot obtain systems with eight invariant straight lines (considering the infinite one and their multiplicities).

***c*)** Assume now $g = 2m \neq 0$/ Then we calculate

$$\text{Coefficient}[R_X^{(2)}(F_1, [P^*(X,Z)]^3), Y^2Z^6] = 1024m^8(cf - f^2 - 2am)^2 = 0$$

from which it results $a = (c-f)f/(2m)$. This condition implies $R_X^{(0)}(F_1, [P^*(X,Z)]^3) = 0$ and

$$R_X^{(1)}(F_1, [P^*(X,Z)]^3) = -[8bm^3 - (2c-5f)(c-f)^2 - 4efm^2]^2(f^3 - 4efm^2 + 8bm^3)^3 Z^{15}/(2m) = 0,$$
$$\Phi = (f^3 - 4efm^2 + 8bm^3)\tilde{\Phi}(y) \neq 0.$$

Therefore it remains to only examine the equality $8bm^3 - (2c-5f)(c-f)^2 - 4efm^2 = 0$, i.e. $b = \left[(2c-5f)(c-f)^2 + 4efm^2\right]/(8m^3)$ which implies

$$R_X^{(2)}(F_1, [P^*(X,Z)]^3) = 24(c-2f)^4(2c-f)^3(c-f)m^2Z^8 = 0,$$
$$\Phi(y) = (c-2f)^3(2c-f)(3c^2 - 6cf + 3f^2 - 4em^2 - 8m^3y)/(8m^3) \neq 0.$$

We detect that the above conditions are equivalent to $f = c \neq 0$ (due to $\Phi \neq 0$) and this gives $g = 2m, a = 0$ and $b = ec/(2m)$. On the other hand for the corresponding systems we get $R_X^{(0)}(\mathcal{G}_2/H, \mathcal{G}_1/H) = 12c^2m^2Z^3 \neq 0$, i.e. we could not have invariant lines of total multiplicity 9. In this case applying an additional transformation $(x,y,t) \mapsto (cx/(2m), (c^2y - 4em^2)/(8m^3), t/c)$ we arrive at the systems

$$\dot{x} = x(1+x), \quad \dot{y} = y + xy - x^3 \tag{3.28}$$

with $H(X,Z) = X^4Z^2(X+Z)$.

Taking into consideration Lemma 2.1 for the above system we calculate: $\mu_0 = \ldots = \mu_7 = \mu_9 = 0,\ \mu_8 = x^8$. By Lemma 2.1 eight finite singular points from nine have gone to infinity and collapsed with the singular point $[0,1,0]$ and the invariant straight lines $L_{1,2,3,4}:\ x = 0,\ \ L_5:\ x+1 = 0,\ \ L_{6,7}: Z = 0$. Therefore we get the configuration given by *Config. 8.50* (see Figure 3.2).

Systems with configuration (3,2,1,1) Considering systems (3.23) for $k = 0$ (see Proposition 3.1) we evaluate equations (2.17) for the direction $x = 0$ (i.e. $U = 1, V = 0$):

$$Eq_1 = \ldots = Eq_8 = 0, \quad Eq_9 = d - 2hW,\ Eq_{10} = a - cW + gW^2. \tag{3.29}$$

It is clear that in the case $h \neq 0$ we could have only one invariant affine line in the direction $x = 0$ and in order to have two such lines the condition $h = 0$ is necessary. So we split our examination in two cases: $h = 0$ and $h \neq 0$.

1) The case $h = 0$**.** By Remark 3.5 the in order to have invariant lines of multiplicity 8 there must be at least one invariant affine line. So the condition $d = 0$ must be satisfied and considering also the condition $k = 0$ we get the following systems

$$\dot{x} = a + cx + gx^2 \equiv P(x), \quad \dot{y} = b + ex + fy + 2mxy + ny^2 - x^3 \equiv Q(x, y) \qquad (3.30)$$

for which $H(X, Z) = Z^2(gX^2 + cXZ + aZ^2)$. Since the degree of H is four we have to find out a common factor of degree three of the polynomials $\mathcal{G}_i/H,\ i = 1, 2, 3$. We calculate

$$\mathcal{G}_1/H \equiv F_1(X, Y, Z), \quad \mathcal{G}_2/H = -nX^6 + ZF_2(X, Y, Z), \quad \mathcal{G}_3/H = 24Z^2[P^*(X, Z)]^3$$

where $P^*(X, Y)$ is the homogenized form of the polynomial P(x) and $F_1(X, Y, Z)$ (respectively $F_2(X, Y, Z)$) is a polynomial of degree 4 (respectively 5) with respect to the variables.

We observe that in the case $n = 0$ we get systems (3.24) for which we already have detected all possible configurations of invariant lines of total multiplicity eight.

We claim that in the case $n \neq 0$ for non-degenerate systems (3.30) the polynomials G_1/H and $P^*(X, Z)$ could not have a common factor depending on X, i.e. the condition $R^{(0)}_X(F_1, P(X, Z)) \neq 0$ holds.

Indeed, since $n \neq 0$ we may assume $n = 1$ due to the rescaling $(x, y, t) \mapsto (x/n, y/n^2, nt)$ and we calculate Coefficient$[R^{(0)}_X(F_1, P^*(X, Z)), Y^6Z^2] = 4g^2$. So supposing that the condition $R^{(0)}_X(F_1, P(X, Z)) = 0$ holds we get $g = 0$. Therefore we have

$$R^{(0)}_X(F_1, P^*(X, Z)), Y^3Z^5] = 2mZ^5[-2cY + (c^2 - cf + 2am)Z][c^3Y^2 + \\ +c^2(cf - 2am)YZ + (a^3 + bc^3 - ac^2e)Z^2] = 0$$

and this implies either $m = 0$ or $c = a = 0$, however in the second case we get degenerate systems. So $m = 0$ and setting $F_1' = F_1|_{g=m=0}$ we calculate

$$R^{(0)}_X(F_1', P^*(X, Z)) = Z^4(2Y - cZ + fZ)(2Y + 2cZ + fZ)\big[c^3Y^2 + c^3fYZ + \\ +(a^3 + bc^3 - ac^2e)Z^2\big] = 0, \ \Phi(y) = c^3y^2 + c^3fy + (a^3 + bc^3 - ac^2e) \neq 0.$$

The contradiction we obtained completes the proof of our claim.

2) The case $h \neq 0$**.** Then we may assume $h = 1$ due to the rescaling $(x, y, t) \mapsto (x/h, y/h^2, ht)$ and therefore the condition $R(0)_W(Eq_9, Eq_{10}) = d^2g - 2cd + 4a = 0$ gives $a = d(2c - dg)/4$. Therefore applying the transformation $(x, y, t) \mapsto (x - d/2, y + (dg - h)/2, t)$ we get the family of systems

$$\dot{x} = gx^2 + 2xy \equiv P(x), \quad \dot{y} = b + fy + ex + ny^2 + lx^2 + 2mxy - x^3 \equiv Q(x, y). \qquad (3.31)$$

For these systems we calculate $H(X, Z) = XZ$ and $\mathcal{G}_1/H = 2X^6 + Zf(X, Y, Z),\ \mathcal{G}_3/H = 24Z^3X^3(gX + 2Y)^4$. We observe that the above polynomials cannot have Z as a common factor. On the other hand evidently the factor $gX + 2Y$ cannot give a line in the direction

$x = 0$. Therefore a common factor of maximum degree of the polynomials $\mathcal{G}_1/H$ and $\mathcal{G}_3/H$ could be only X^3. Taking into consideration that for systems (3.31) the polynomial H has degree two (but should be seven) we conclude that in this case we cannot obtain systems with invariant lines of total multiplicity eight.

Systems with configuration (2,2,2,1). So we consider systems (3.23) for $k = n = 0$ and $k = m = n - h = 0$ and in both these case we solve the equations (2.17) setting $U = 1,\ V = 0$:

$$Eq_1 = \ldots = Eq_8 = 0, \quad Eq_9 = d - 2hW,\ Eq_{10} = a - cW + gW^2.$$

Thus we arrive at the same equations (3.29) as in the case of systems (3.23) for $k = 0$. Since setting the condition $n = 0$ we get a subfamily of the systems which are already examined, we conclude that in the case under examination we could not have invariant lines of total multiplicity eight.

The above results lead as to the following proposition:

Proposition 3.2. *Non-degenerate systems* (3.24) *possess invariant lines of total multiplicity eight if and only if one of the following sets of conditions hold:*

$$g = m = a = 0, \quad f = -2c, \quad c \neq 0; \tag{3.32}$$

$$g = m = 0, \quad f = c, \quad e = 3a^2/c^2,\ a\ c \neq 0; \tag{3.33}$$

$$a = 0, \quad g = 2m, \quad f = c, \quad b = ec/(2m),\ m\ c \neq 0. \tag{3.34}$$

3.2.2.2. Construction of the normal form with cubic homogeneities $(x^3, -x^3 + x^2y)$

. In this case due to the translation of the origin of coordinates to the point $(x_0, y_0) = (-m, -l - 3m)$ we can consider $l = m = 0$ and hence we get the cubic systems

$$\dot{x} = a + cx + dy + gx^2 + 2hxy + ky^2 + x^3, \quad \dot{y} = b + ex + fy + ny^2 + x^2y - x^3. \tag{3.35}$$

Next we force the necessary conditions given by Remark 3.1 which correspond to the type of configuration $(3, 2, 1, 1)$. We calculate $\mathcal{K}_4 = -x^2[(2h+n)x+ky]/9 = 0,\ \mathcal{K}_5 = -1200h^2x^4 = 0$ which imply $k = h = n = 0$. Herein calculations lead to

$$\mathcal{K}_6 = -40(96a - 68cg + 1739dg - 136fg)x^{11}/27 - 6560dgx^{10}y/27.$$

It is evident that the condition $\mathcal{K}_6 = 0$ implies

$$d \cdot g = 0, \quad 24a - 17cg - 34fg = 0. \tag{3.36}$$

Considering systems (3.35) for $k = n = h = 0$ we calculate the equations (2.17) for the direction $x = 0$ (i.e $U = 1,\ V = 0$): $Eq_1 = \ldots = Eq_8 = 0, \quad Eq_9 = d,\ Eq_{10} = a -$

$cW + gW^2 - W^3$. On the other hand we detect that for the corresponding systems we have $\mathcal{G}_1/H|_{Z=0} = X^8$, therefore all three polynomials $\mathcal{G}_i/H,\ i = 1, 2, 3$ could not have as a common factor Z. So we shall examine only the conditions given by the resultants with respect to X.

Clearly the condition $d = 0$ is necessary for the existence of at least one invariant affine line and in this case the systems possess exactly three such lines (which could be real or complex, distinct or coinciding). Therefore we arrive at the systems

$$\dot{x} = a + cx + gx^2 + x^3 \equiv P(x) \quad \dot{y} = b + ex + fy + x^2y - x^3 \equiv Q(x, y). \tag{3.37}$$

For these systems we calculate

$$\mathcal{G}_1/H \equiv F_1(X, Y, Z), \quad \mathcal{G}_2/H = P^\star(X, Z)F_2(X, Y, Z), \quad \mathcal{G}_3/H = 24[P^\star(X, Z)]^3,$$
$$H(X, Z) = X^3 + gX^2Z + cXZ^2 + aZ^3 \equiv P^\star(X, Z).$$

Since the polynomial H has degree 3 we conclude that we need to find out a common factor of degree four of the polynomials $\mathcal{G}_i/H,\ i = 1, 2, 3$ depending on X, i.e. the condition $R^{(0)}_X(F_1, [P^\star]^3) = R^{(1)}_X(F_1, [P^\star]^3) = R^{(2)}_X(F_1, [P^\star]^3) = R^{(3)}_X(F_1, [P^\star]^3) = 0$ has to be satisfied.

We calculate Coefficient$[R^{(3)}_X(F_1, [P^\star]^3), Y^4Z^8] = (c-f)^4 = 0$ and this condition is equivalent to $f = c$ which leads to Coefficient$[R^{(1)}_X(F_1, [P^\star]^3), Y^8Z^{24}] = 256(a - cg)^8 = 0$, i.e. $a = cg$. Then considering (3.36) we get $cg = 0$ and we divide our examination in two cases: $g = 0$ and $c = 0,\ g \neq 0$.

1) First we examine the case $g = 0$. Then $\varphi(y) = (b + cy)\tilde{\varphi} \neq 0$ and $R^{(1)}_X(F_1, [P^\star]^3) = 64c^5e^2\tilde{\varphi}^3Z^{32} = 0$ which gives $ec = 0$. If $c = 0$ then we get $R^{(2)}_X(F_1, [P^\star]^3) = -128b^7Z^{21} = 0$, i.e. $b = 0$ which contradicts with $\varphi = b^3 \neq 0$.

So $e = 0,\ c \neq 0$ and in this case we have $\varphi(y) = (b^2+c^3)(b+cy) \neq 0$ and $R^{(2)}_X(F_1, [P^\star]^3) = -128b(b^2+c^3)^3Z^{21} = 0$, i.e. $b = 0$. However this condition implies $R^{(3)}_X(F_1, [P^\star]^3) = 64c^6Z^{12} \neq 0$ (since $c \neq 0$). Therefore if $e = 0$ and $c \neq 0$ we could not obtain systems with invariant lines of total multiplicity eight.

2) Now we consider the case $c = 0,\ g \neq 0$. Then $R^{(0)}_X(F_1, [P^\star]^3) = R^{(1)}_X(F_1, [P^\star]^3) = 0$ and

$$R^{(2)}_X(F_1, [P^\star]^3) = -32b^5(2b - eg - g^3)^2Z^{21} = 0.$$

In addition $\varphi(y) = b^2(b-eg+g^3+g^2y) \neq 0$ and hence the above relation gives $2b-eg-g^3 = 0$, i.e. $b = g(e+g^2)/2$. Then $R^{(3)}_X(F_1, [P^\star]^3) = (e+g^2)^5(e+3g^2)Z^{12} = 0$ which implies $e = -3g^2$ since $\varphi(y) = g^3(e + g^2)^2(3g^2 - e + 2gy)/8 \neq 0$. Therefore we arrive at the systems

$$\dot{x} = x^2(g + x), \quad \dot{y} = -g^3 - 3g^2x + x^2y - x^3$$

and since $g \neq 0$ due to the rescaling $(x, y, t) \mapsto (gx, gy, t/g^2)$ the above systems become of the form

$$\dot{x} = x^2(1 + x), \quad \dot{y} = -1 - 3x + x^2y - x^3 \tag{3.38}$$

with $H(X,Z)=X^3(X+Z)^4$.

Taking into consideration Lemma 2.1 for the above system we calculate: $\mu_0=\ldots=\mu_7=0$, $\mu_8=x^8$ and hence eight finite singular points from nine have gone to infinity and collapsed with the singular point $[0,1,0]$ located on the "end" of the invariant line $x=0$. Additionally we determine that system (3.38) possesses the finite singular point $(-1,-3)$ and the invariant straight lines: $L_{1,2,3}: \ x=0$, $L_{4,\ldots 7}: \ x+1=0$. This leads to the configuration *Config. 8.51.*

Proposition 3.3. *Non-degenerate systems* (3.35) *possess invariant lines of total multiplicity eight if and only if the following set of conditions hold:*

$$k=n=h=a=c=f=0, \quad b=-g^3, \quad e=-3g^2, \quad g\neq 0. \tag{3.39}$$

3.2.3. Invariant criteria for the realization of the configurations with exactly one infinite singularity

According to Lemma 2.2 the conditions $\mathcal{D}_1=\mathcal{D}_2=\mathcal{D}_3=0$ are necessary and sufficient for a cubic system to have exactly one infinite singularity, namely it is determined by the unique factor of degree four of $C_3(x,y)$. After a linear transformation such a cubic system could be brought to the form (3.19). By Lemma 3.1 these systems could have one of the following two cubic homogeneities: $(0,-x^3)$ and $\left(x^3, x^2y-x^3\right)$. Moreover, according to the same lemma, these two cases of homogeneities are distinguished by the invariant polynomials $\mathcal{V}_1$ and $\mathcal{V}_5$. Therefore we consider systems (3.23) and respectively (3.35) and in what follows we find out the invariant conditions which are equivalent to the conditions given by Propositions 3.2 and 3.3, respectively.

1) **Conditions for systems** (3.23) According to Proposition 3.2 the condition $k=h=n=0$ which is equivalent to $\mathcal{L}_2=0$ (see Proposition 3.1) is necessary for systems (3.23) to have invariant lines of total multiplicity eight. For the corresponding systems we calculate:

$$N_{23}=m^2x^6, \ W_1=-6(g+m)x^5=0, \ W_2=12(72d+13g^2-54gm+53m^2)x^6.$$

a) *Conditions* (3.32). It is evident that $N_{23}=W_1=W_2=0$ is equivalent to $m=g=d=0$. Then we calculate $W_3=8(2c+f)x^4$ which is equivalent to $f=-2c$ and this condition implies $W_4=108ax^3$. Therefore the condition $W_4=0$ gives $a=0$ and we arrive at non-degenerate systems (where $\mu_8=-2c^4x^8\neq 0$) which, as it was shown earlier (see page 97), via a transformation could be brought to systems (3.26).

Thus if for systems (3.23) the conditions $\mathcal{L}_2=N_{23}=W_1=W_2=W_3=W_4=0$, $\mu_8\neq 0$ hold then we get the configuration *Config. 8.48.*

b) *Conditions* (3.33). As it was shown above So the condition $N_{23}=W_1=W_2=0$ implies $m=g=d=0$ and then $N_{16}=12(f-c)x^4$. So the condition $f=c$ is given by

$N_{16} = 0$. In this case we obtain $\mu_8 = c^4x^8$ and $W_5 = 12(3a^2 - c^2e)x^2$. Therefore the condition $W_5 = 0$ leads to $e = 3a^2/c^2$ for $c \neq 0$, i.e. $\mu_8 \neq 0$. For the corresponding non-degenerate systems the condition $a \neq 0$ does not allow to have invariant lines of total multiplicitynine and this condition is equivalent to $W_6 = -12ax^3 \neq 0$. Then due to a transformation the corresponding systems become of the form (3.27).

$W_6 = -12ax^3 \neq 0$ is satisfied, i.e.

So if for systems (3.23) the conditions $\mathcal{L}_2 = N_{23} = W_1 = W_2 = N_{16} = W_5 = 0$, $\mu_8 W_6 \neq 0$ hold then we obtain *Config. 8.49.*

c) *Conditions* (3.34). According to this set of conditions we have $m \neq 0$, i.e. $N_{23} \neq 0$. We calculate $N_3 = 12(2m - g)x^5$ and clearly the condition $g = 2m$ is equivalent to $N_3 = 0$. In this case we obtain $W_7 = 48dmx^3$ and hence the condition $d = 0$ is governed by the polynomial W_7. So considering the conditions $g = 2m$ and $d = 0$ we get $W_8 = 1728(c - f)mx^5$ and since $m \neq 0$ the condition $W_8 = 0$ is equivalent to $f = c$. Then we obtain $W_9 = -864am^3x^2$ and evidently that the condition $W_9 = 0$ gives $a = 0$. Setting $a = 0$ we get $W_{10} = 72m(ce - 2bm)x^3$ and due to $m \neq 0$ it is clear that the condition $b = ce/(2m)$ is equivalent to $W_{10} = 0$. Assuming that this condition is fulfilled, for the corresponding systems we get $\mu_8 = c^4x^8 \neq 0$ and applying a transformation (see page 100) these systems become of the form (3.28).

Thus if for systems (3.23) the conditions $\mathcal{L}_2 = N_3 = W_7 = W_8 = W_9 = W_{10} = 0$, $\mu_8 N_{23} \neq 0$ hold then we get the configuration *Config. 8.50.*

2) Conditions for systems (3.35). It was proved that the condition $k = h = n = 0$ is equivalent to $\mathcal{K}_4 = \mathcal{K}_5 = 0$ (see the page 102). In this case we get $\mathcal{K}_8 = 88dx^4/9 = 0$ which implies $d = 0$ and then we calculate

$$\mathcal{K}_9 = 20(f - c)x^6/9, \quad N_2 = -(c + f)x^4/3, \quad \mathcal{K}_6 = 160(17cg + 34fg - 24a)x^{11}/27.$$

It is evident that the condition $\mathcal{K}_9 = N_2 = \mathcal{K}_6 = 0$ gives $c = f = a = 0$. Then we obtain $W_{11} = -9216(b + g^3)x^{11}$ and therefore $b = -g^3$ is equivalent to $W_{11} = 0$. Setting this condition we calculate $W_{12} = 20g(e + 3g^2)x^{11}/9$ and $\mu_8 = g^8x^8 \neq 0$ (otherwise we get degenerate systems). So the condition $e = -3g^2$ is equivalent to $W_{12} = 0$ and then due to a transformation (see page 104) we arrive at the system (3.38).

Thus if for systems (3.35) the conditions $\mathcal{K}_4 = \mathcal{K}_5 = \mathcal{K}_8 = \mathcal{K}_9 = N_2 = \mathcal{K}_6 = W_{11} = W_{12} = 0$, $\mu_8 \neq 0$ hold then we have *Config. 8.51.*

3.2.4. Perturbations of canonical forms

To end the proof of the Main Theorem C it remains to construct for the normal forms given by this theorem the corresponding perturbations, which prove that the respective invariant straight lines have the indicated multiplicities. In this section we construct such perturbations and for each configuration *Configs. 8.j*, $j = 48, \ldots, 51$ we give:

- the corresponding normal form and its invariant straight lines;
- the respective perturbed normal form and its invariant straight lines;
- the configuration *Config.* $8.j_\varepsilon$, $j = 48, \ldots, 51$ which corresponds to the perturbed systems.

Config. 8.48: $\quad \dot{x} = x, \quad \dot{y} = -2y - x^3;$

Invariant lines: $\quad L_1 : \ x = 0, \quad L_{2,\ldots 7} : \ Z = 0;$

Config. 8.48_ε: $$\begin{cases} \dot{x} = x(1 + x^2\varepsilon - 6y^2\varepsilon^3), \\ \dot{y} = -x^3 - 2y - 4y^3\varepsilon^3; \end{cases}$$

Invariant lines: $$\begin{cases} L_1 = x, \ L_{2,3} = x^2\varepsilon - i + (2-2i)xy\varepsilon^2 - 2iy^2\varepsilon^3, \\ L_{4,5} = x^2\varepsilon + i + (2+2i)xy\varepsilon^2 + 2iy^2\varepsilon^3, \\ L_{6,7} = 1 + 2x^2\varepsilon - 4xy\varepsilon^2 + 2y^2\varepsilon^3. \end{cases}$$

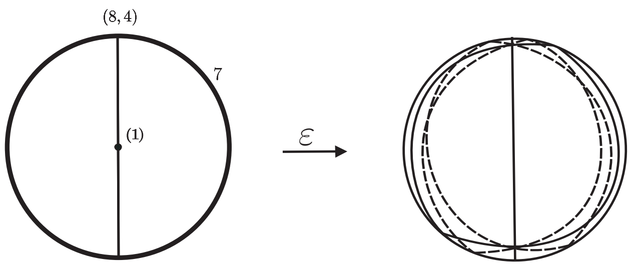

Fig. 3.8. Perturbation of normal form corresponding to the configuration *Config. 8.48*

Config. 8.49: $\quad \dot{x} = x, \quad \dot{y} = y - x^2 - x^3;$

Invariant lines: $\quad L_{1,2,3} : \ x = 0, \quad L_{4,\ldots 7} : \ Z = 0;$

Config. 8.49_ε: $$\begin{cases} \dot{x} = x(9 - 6x\varepsilon + 4x\varepsilon^2)(9 + 6x\varepsilon - 10x\varepsilon^2 + 4x\varepsilon^3)/81, \\ \dot{y} = \big[3x^3(1-\varepsilon)(2\varepsilon-3)^3 + 3x^2(3-2\varepsilon)^2\big(y(\varepsilon-1)\varepsilon^2 - 3\big) + \\ \qquad +3xy\varepsilon^2(2\varepsilon-3)[6 + y(\varepsilon-1)\varepsilon^2] + y(9+y\varepsilon^3)\big(9 + y(\varepsilon-1)\varepsilon^3\big)\big]/81; \end{cases}$$

Invariant lines: $$\begin{cases} L_1 = x, L_2 = 2x\varepsilon - y\varepsilon^2 - 3x, L_3 = 2x\varepsilon + y\varepsilon^2 - 3x, L_4 = 9 - 6x\varepsilon + 4x\varepsilon^2, \\ L_5 = 9 + 2x\varepsilon(3 - 5\varepsilon + 2\varepsilon^2), \ L_6 = 9 + y\varepsilon^3 + x\varepsilon(2\varepsilon - 3), \\ L_7 = 9 + y(\varepsilon-1)\varepsilon^3 + x\varepsilon(3 - 5\varepsilon + 2\varepsilon^2). \end{cases}$$

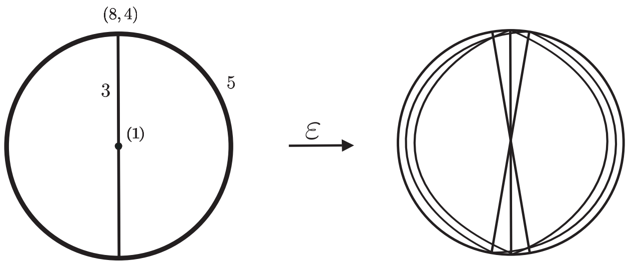

Fig. 3.9. Perturbation of normal form corresponding to the configuration *Config. 8.49*

Config. 8.50: $\dot{x} = x(1+x), \quad \dot{y} = y + xy - x^3;$

Invariant lines: $L_{1,2,3,4}: \ x = 0, \quad L_5 : x+1 = 0, \quad L_{6,7}: \ Z = 0;$

Config. 8.50$_\varepsilon$:
$$\begin{cases} \dot{x} = x(1-x\varepsilon)(1+x+x\varepsilon-2y\varepsilon), \\ \dot{y} = y + xy - x^3 - 2y^2\varepsilon - x^2y(3\varepsilon-4)\varepsilon + 6xy^2(\varepsilon-1)\varepsilon^2 - 4y^3(\varepsilon-1)\varepsilon^3; \end{cases}$$

Invariant lines:
$$\begin{cases} L_1 = x, \ L_2 = x - 2y\varepsilon, \ L_3 = x - y\varepsilon, \ L_4 = x - 2y\varepsilon + 2y\varepsilon^2, \\ L_5 = 1 + x - x\varepsilon - 2y\varepsilon + 2y\varepsilon^2, \ L_6 = x\varepsilon - 1, \ L_7 = 1 + x\varepsilon - 2y\varepsilon^2. \end{cases}$$

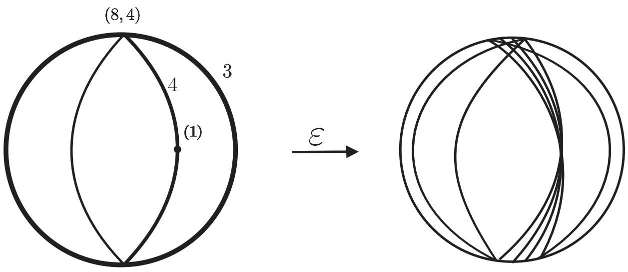

Fig. 3.10. Perturbation of normal form corresponding to the configuration *Config. 8.50*

Config. 8.51: $\dot{x} = x^2(1+x), \quad \dot{y} = -1 - 3x + x^2y - x^3;$

Invariant lines: $L_{1,2,3}: \ x = 0, \quad L_{4,5,6,7}: \ x+1 = 0;$

Config. 8.51$_\varepsilon$:
$$\begin{cases} \dot{x} = x(1+x)(x - \varepsilon - 2x\varepsilon), \\ \dot{y} = -\big[1 + 3x + x^3 - x^2y - 4y^3(\varepsilon-1)^3\varepsilon^4 - y(\varepsilon-1)\varepsilon(3+2\varepsilon) + \\ \quad +2y^2(\varepsilon-1)^2\varepsilon^2(1+3\varepsilon) - xy(\varepsilon-1)\varepsilon(7 - 9\varepsilon - 24\varepsilon^2 + 36\varepsilon^3) + \\ \quad +3x^3(\varepsilon-1)\varepsilon(3 - 4\varepsilon - 12\varepsilon^2 + 54\varepsilon^3 - 72\varepsilon^4 + 36\varepsilon^5) + \\ \quad +2xy^2(\varepsilon-1)^2\varepsilon^2(1 + 3\varepsilon - 18\varepsilon^2 + 18\varepsilon^3) + \\ \quad +3x(\varepsilon-1)\varepsilon(2+3\varepsilon) + x^2 3\varepsilon(3\varepsilon-2)(1 - 6\varepsilon^2 + 6\varepsilon^3) - \\ \quad -x^2y\varepsilon(18\varepsilon + 12\varepsilon^2 - 168\varepsilon^3 + 360\varepsilon^4 - 324\varepsilon^5 + 108\varepsilon^6 - 7)\big]/(\varepsilon-1)^2; \end{cases}$$

Invariant lines: $\begin{cases} L_1 = x, \ L_2 = \varepsilon + x(2\varepsilon - 1), \ L_3 = 1 - 2y(\varepsilon - 1)\varepsilon + x(1 - 6\varepsilon + 6\varepsilon^2), \\ L_4 = 1 + x, \ L_5 = 1 - 2y(\varepsilon - 1)\varepsilon + x(1 - 6\varepsilon + 6\varepsilon^2), \\ L_6 = 1 - y(\varepsilon - 1)\varepsilon + x(1 - 3\varepsilon + 3\varepsilon^2), L_7 = 1 - 2y(\varepsilon - 1)\varepsilon^2 + x(1 - 6\varepsilon^2 + 6\varepsilon^3). \end{cases}$

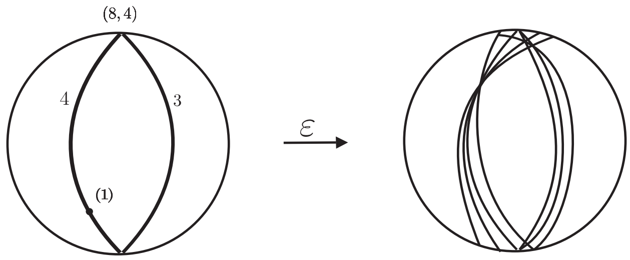

Fig. 3.11. Perturbation of normal form corresponding to the configuration *Config. 8.51*

3.3. Conclusions on Chapter 3

We remark that in Chapter 3 we use for the first time the perturbation of the cubic systems with multiple lines in order to prove that these systems indeed possess invariant lines of indicated multiplicity. It arise the question: why this multiplicity could not be deduced applying the respective definitions and propositions from Chapter 2?

More precisely, according to [43, Definition 5.1] each line ($L = ux + vy + w = 0$) of multiplicity k divide $\mathcal{E}_1(\mathbb{X})$ ($\equiv -\mathcal{G}_1$) and according to [43, Main Theorem] it follows the existence of a perturbation of vector filed $\mathbb{X}$ which implies the appearance in the vicinity of the line $L = 0$ exactly k distinct invariant straight lines. However this is valid for each line separately and this does not guaranty that there exists such a perturbation which implies the appearance of the corresponding numbers of lines in the vicinities of several lines simultaneously. So the constructed perturbations are necessary to complete the proofs of Main Theorems B and C and hence, each of the configurations *Configs. 8.18-8.22* and *Configs. 8.48-8.51* indeed contain invariant lines of total multiplicity eight. We note that we have constructed the corresponding perturbations also for the family of systems in $\mathbb{CSL}_8$ with two distinct infinite singularities (see the proof of Main Theorem D).

The interesting fact is that all 9 canonical systems constructed in this chapter for systems in $\mathbb{CSL}_8$ with either three or exactly one infinite singular points are with constant coefficients.

Finally we underline that the configurations given by *Config. 8.48, 8.49* and *Config. 8.51* were also detected by Şubă and Vacaraş in [130–132]. More precisely, the configuration *Config. 8.48* possesses the infinite line of the multiplicity 7 and it was proved in [130] that this multiplicity is maximal for a cubic system. But it is important to underline that in contrast with these authors in Chapter 4 necessary and sufficient conditions for the realization of the configurations given by *Config.8.48, 8.49* and *Config.8.51* were determined.

The results presented in Chapter 3 were published in [22, 25].

4. CUBIC SYSTEMS WITH INVARIANT LINES OF TOTAL MULTIPLICITY EIGHT AND TWO DISTINCT INFINITE SINGULARITIES

Taking into account Lemma 2.2 we split the family of cubic systems having 2 distinct infinite singularities in three subfamilies which infinite singularities are determined by the following factors of the polynomial $C_3(x, y)$: 1) two double real factors; 2) two double complex factors and 3) one triple and one simple real factors. For each one of these three subfamilies the proof of the corresponding theorem proceeds in 4 steps described in Paragraph 2.1.2.

In this chapter we prove the following theorem ([17, 29]):

Main Theorem D. *Assume that a non-degenerate cubic system (i.e.* $\sum_{i=0}^{9}\mu_i^2 \neq 0$*) possesses invariant straight lines of total multiplicity 8, including the line at infinity with its own multiplicity. In addition we assume that this system has two distinct infinite singularities, i.e. the conditions* $\mathcal{D}_1 = \mathcal{D}_3 = 0$ *and* $\mathcal{D}_2 \neq 0$ *hold. Then:*

I. *This system could not have the infinite singularities defined by two double factors of the invariant polynomial* $C_3(x, y)$.

II. *The system has the infinite singularities defined by one triple and one simple real factors of* $C_3(x, y)$ *(i.e.* $\mathcal{D}_1 = \mathcal{D}_3 = \mathcal{D}_4 = 0$ *and* $\mathcal{D}_2 \neq 0$*) and could possess only one of the 25 possible configurations Config. 8.23 – Config. 8.47 of invariant lines given in Figure 4.1.*

III. *This system possesses the specific configuration Config. 8.j* ($j \in \{23, 24, \dots, 47\}$) *if and only if the corresponding conditions included below are fulfilled. Moreover it can be brought via an affine transformation and time rescaling to the canonical form, written below:*

- *Config. 8.23* $\Leftrightarrow N_2N_3 \neq 0, \mathcal{V}_1 = \mathcal{V}_3 = \mathcal{K}_5 = N_1 = N_4 = N_5 = N_6 = N_7 = 0$:
$$\begin{cases} \dot{x} = (x-1)x(1+x), \\ \dot{y} = x - y + x^2 + 3xy; \end{cases}$$

- *Config. 8.24 - 8.27* $\Leftrightarrow N_2 \neq 0, N_3 = 0,\ \mathcal{V}_1 = \mathcal{V}_3 = \mathcal{K}_5 = N_1 = N_4 = N_6 = N_8 = 0, N_9 \neq 0$:
$$\begin{cases} \dot{x} = x(r + 2x + x^2), \\ \dot{y} = (r + 2x)y,\ r(9r-8) \neq 0; \end{cases} \qquad \begin{cases} \text{Config.8.24} \Leftrightarrow N_{11} < 0\ (r < 0); \\ \text{Config.8.25} \Leftrightarrow N_{10} > 0, N_{11} > 0\ (0 < r < 1); \\ \text{Config.8.26} \Leftrightarrow N_{10} = 0\ (r = 1); \\ \text{Config.8.27} \Leftrightarrow N_{10} < 0\ (r > 1); \end{cases}$$

- *Config. 8.28 - 8.30* $\Leftrightarrow N_2 \neq 0, N_3 = 0,\ \mathcal{V}_1 = \mathcal{V}_3 = \mathcal{K}_5 = N_1 = N_5 = N_8 = N_{12} = 0, N_{13} \neq 0$:
$$\begin{cases} \dot{x} = x(r - 2x + x^2), (9r-8) \neq 0 \\ \dot{y} = 2y(x - r), r(r-1) \neq 0; \end{cases} \qquad \begin{cases} \text{Config.8.28} \Leftrightarrow N_{15} < 0\ (r < 0); \\ \text{Config.8.29} \Leftrightarrow N_{14} < 0, N_{15} > 0\ (0 < r < 1); \\ \text{Config.8.30} \Leftrightarrow N_{14} > 0\ (r > 1); \end{cases}$$

- *Config. 8.31, 8.32* $\Leftrightarrow N_2 = N_3 = \mathcal{V}_1 = \mathcal{V}_3 = \mathcal{K}_5 = N_1 = N_{17} = N_{18} = 0,\ N_{10}N_{16} \neq 0$:

$$\begin{cases}\dot{x} = x(r+x^2),\\ \dot{y} = x-2ry,\ r\in\{-1,1\};\end{cases}\quad \begin{cases}Config. 8.31 \Leftrightarrow N_{10}<0\ (r=-1);\\ Config. 8.33 \Leftrightarrow N_{10}>0,\ (r=1);\end{cases}$$

- *Config. 8.33* $\Leftrightarrow N_2=N_3=\mathcal{V}_1=\mathcal{V}_3=\mathcal{K}_5=N_1=N_{10}=N_{17}=N_{18}=0, N_{16}\neq 0$: $\begin{cases}\dot{x}=x^3,\\ \dot{y}=1+x;\end{cases}$

- *Config. 8.34 - 8.38* $\Leftrightarrow N_2=N_3=\mathcal{V}_1=\mathcal{V}_3=\mathcal{K}_5=N_1=N_{16}=N_{19}=0,\ N_{18}\neq 0$:

$$\begin{cases}\dot{x} = x(r+x+x^2),\\ \dot{y} = 1+ry, (9r-2)\neq 0;\end{cases}\quad \begin{cases}Config.\ 8.34 \Leftrightarrow N_{21}<0\ (r<0);\\ Config.\ 8.35 \Leftrightarrow N_{20}>0, N_{21}>0\ (0<r<1/4);\\ Config.\ 8.36 \Leftrightarrow N_{20}=0\ (r=1/4);\\ Config.\ 8.37 \Leftrightarrow N_{20}<0\ (r>1/4);\\ Config.\ 8.38 \Leftrightarrow N_{21}=0\ (r=0);\end{cases}$$

- *Config.8.39, 8.40* $\Leftrightarrow \mathcal{V}_1=\mathcal{L}_1=\mathcal{L}_2=N_{22}=N_{23}=N_{24}=0,\ \mathcal{V}_3\mathcal{K}_6\neq 0$:

$$\begin{cases}\dot{x} = x(r+x+x^2),\\ \dot{y} = (r+2x+3x^2)y;\end{cases}\quad \begin{cases}Config.\ 8.39 \Leftrightarrow \mu_6<0\ (r<1/4);\\ Config.\ 8.40 \Leftrightarrow \mu_6>0\ (r>1/4);\end{cases}$$

- *Config.8.41– 8.43* $\Leftrightarrow \mathcal{V}_1=\mathcal{L}_1=\mathcal{L}_2=N_{22}=N_{23}=\mathcal{K}_6=0,\ \mathcal{V}_3N_{24}\neq 0$:

$$\begin{cases}\dot{x} = x(r+x^2),\\ \dot{y} = 1+ry+3x^2y;\end{cases}\quad \begin{cases}Config.\ 8.41 \Leftrightarrow \mu_6<0\ (r<0);\\ Config.\ 8.42 \Leftrightarrow \mu_6=0\ (r=0);\\ Config.\ 8.43 \Leftrightarrow \mu_6>0\ (r>0);\end{cases}$$

- *Config.8.44–8.47* $\Leftrightarrow \mathcal{V}_5=\mathcal{U}_2=\mathcal{K}_4=\mathcal{K}_5=\mathcal{K}_6=N_{24}=N_{25}=N_{26}=N_{27}=0$, $\mathcal{V}_1\mathcal{V}_3\neq 0$:

$$\begin{cases}\dot{x} = x(1+x)[r+2+(r+1)x],\\ \dot{y} = [r+2+(3+2r)x+rx^2]y;\end{cases}\quad \begin{cases}Config.\ 8.44 \Leftrightarrow \mu_6<0\ (-2<r<-1);\\ Config.\ 8.45 \Leftrightarrow \mu_6>0, N_{28}<0\ (r<-2);\\ Config.\ 8.46 \Leftrightarrow \mu_6>0, N_{28}>0\ (r>-1);\\ Config.\ 8.47 \Leftrightarrow \mu_6=0\ (r=-1).\end{cases}$$

We have proved in [26] that systems in $\mathbb{CSL}_8$ could not have the infinite singularities defined by two double factors of the invariant polynomial $C_3(x,y)$. So for the further investigation we consider only cubic systems with infinite singularities defined by one triple and one simple real factors of $C(x,y)$ and we prove the above theorem taking into consideration the steps defined in Paragraph 1.2.6. (see page 50).

According to Lemma 2.2 in this case we consider the following family of cubic systems:

$$\begin{aligned}x' &= a+cx+dy+gx^2+2hxy+ky^2+(u+1)x^3+vx^2y+rxy^2,\\ y' &= b+ex+fy+lx^2+2mxy+ny^2+ux^2y+vxy^2+ry^3\end{aligned} \tag{4.1}$$

with $C_3=x^3y$. Hence, the infinite singular points are situated at the “ends” of the straight lines $x=0$ and $y=0$.

The cubic homogeneous systems in the case under consideration were constructed in

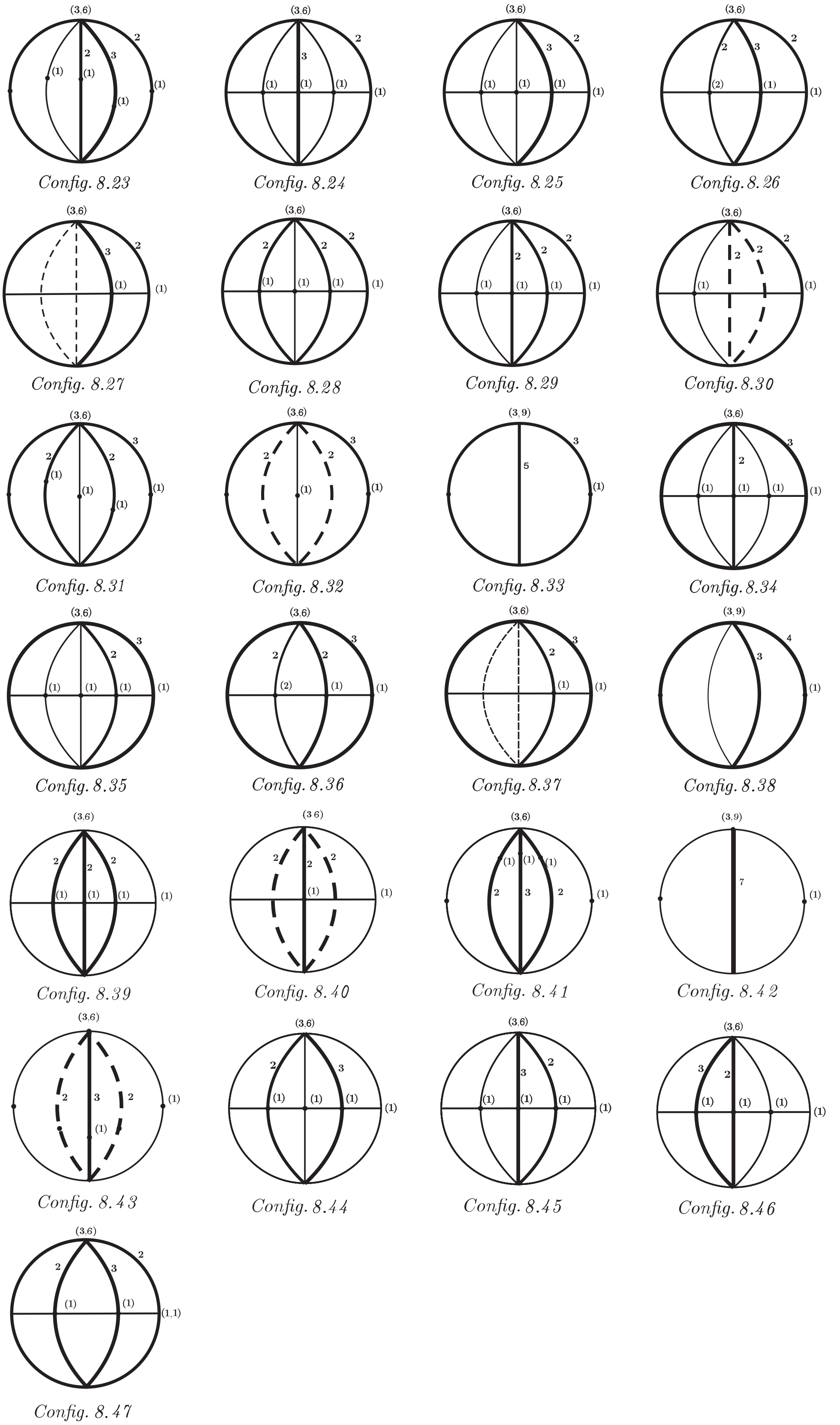

Fig. 4.1. Configurations of invariant lines for systems in $\mathbb{CSL}_8$ with 2 $\mathbb{ISP}s$

[17, Proposition 9]. More precisely considering the proof of Proposition 9 [17] we have the next result.

Lemma 4.1 ([17]). *Assume that for a cubic homogeneous system the conditions $\mathcal{D}_1 = \mathcal{D}_3 = \mathcal{D}_4 = 0$ and $\mathcal{D}_2 \neq 0$ hold. Then via a linear transformation and time rescaling this system can be brought to one of the canonical form $(\tilde{P}_3, \tilde{Q}_3)$ indicated bellow if and only if the corresponding conditions are satisfied. Moreover the cubic systems with these homogeneities could have only the configurations of invariant lines of the type given next to each homogeneity:*

- $\mathcal{V}_1 = \mathcal{V}_3 = 0 \;\Rightarrow\; (\tilde{P}_3, \tilde{Q}_3) = (x^3, 0) \;\Rightarrow\; (3,3,1), (3,2,1,1), (2,2,2,1)$;
- $\mathcal{V}_1 = 0,\ \mathcal{V}_3 \neq 0 \;\Rightarrow\; (\tilde{P}_3, \tilde{Q}_3) = (x^3, 3x^2y) \;\Rightarrow\; (3,3,1)$;
- $\mathcal{V}_1 \neq 0,\ \mathcal{V}_3 = 0 \;\Rightarrow\; (\tilde{P}_3, \tilde{Q}_3) = (2x^3, 3x^2y) \;\Rightarrow\; (2,2,2,1)$;
- $\mathcal{V}_1\mathcal{V}_3 \neq 0,\ \mathcal{V}_5 = \mathcal{U}_2 = 0 \;\Rightarrow\; (\tilde{P}_3, \tilde{Q}_3) = \big((u+1)x^3, ux^2y\big),\ u(u+3)(2u+3) \neq 0 \Rightarrow (3,2,1,1)$.

4.1. Construction of canonical forms and of the corresponding configurations of invariant lines

We have showed in [17] that systems (4.1) possessing the homogeneity $(2x^3, 3x^2y)$ could not have invariant lines of total multiplicity eight and thus, in the further investigation we consider only the following three cubic parts: $(x^3, 0)$, $(x^3, 3x^2y)$ and $\big((u+1)x^3, ux^2y\big)$ (for $u(u+3)(2u+3) \neq 0$).

4.1.1. Systems with cubic homogeneous parts $(x^3, 0)$

In this case, considering (4.1) via a translation of the origin of coordinates we can consider $g = 0$ and hence we get the cubic systems

$$\dot{x} = a + cx + x^3 + dy + 2hxy + ky^2, \quad \dot{y} = b + ex + lx^2 + fy + 2mxy + ny^2 \tag{4.2}$$

for which we have $H(X, Y, Z) = Z$ (see Notation 2.2).

Now we force the necessary conditions given in Remark 3.1 (p. 81) which correspond to each type of configuration. We claim that if any of the conditions (a_1), (a_2) or (a_3) are satisfied for a system (4.2) then $k = h = n = 0$ and this condition is equivalent to $\mathcal{K}_5 = 0$. We divide the proof of this claim in three subcases defined by (a_1)–(a_3).

(a_1). For systems (4.2) we calculate: $\mathcal{L}_1 = 0$ and Coefficient$[\mathcal{L}_2, xy] = -20736(12h^2 + 7km - 6hn + 3n^2) = 0$, Coefficient$[\mathcal{K}_1, y^2] = 3967 \cdot 2^{18}3^9 5^4 7^3 19 k^6 = 0$. Therefore we get $k = 0$ and as the discriminant of the binary form $4h^2 - 2hn + n^2$ is negative we obtain $h = n = 0$ (and this implies $\mathcal{L}_2 = \mathcal{K}_1 = 0$).

(a_2). In the same manner in the case of the configuration $(3,2,1,1)$ we determine $\mathcal{K}_4 = \mathcal{K}_6 = 0$ and $\mathcal{K}_5 = -180m(h-n)x^4 + 60(4h^2 - 3km - 2hn + n^2)x^3y - 240k^2xy^3$. From $\mathcal{K}_5 = 0$

it results $k=0$ and we get the same binary form $4h^2-2hn+n^2$ which leads to $h=n=0$. Consequently $\mathcal{K}_5=0$ if and only if $k=h=n=0$.

(a_3). We calculate $\mathcal{K}_4=0$ and Coefficient$[\mathcal{L}_2, x^2y^7]=2k^3=0$, i.e. $k=0$. Then calculations yield

$$\text{Coefficient}[\mathcal{K}_2, x^5y^4]=-2n(h-n)^2=0$$

and

$$\text{Coefficient}[\mathcal{K}_8, x^3y]=2(4h^2+14hn+n^2)=0.$$

Evidently we obtain $h=n=0$ (then $\mathcal{K}_2=\mathcal{K}_8=0$) and this completes the proof of the claim.

Remark 4.1. *Since for systems* (4.1) *the condition* $k=h=n=0$ *is equivalent to* $\mathcal{K}_5=0$ *we assume this condition to be fulfilled.*

We begin with the examination of the direction $x=0$ ($U=1,\ V=0$). So, considering (2.17) and Remark 2.1 for systems (4.2) we have:

$$Eq_9=d,\quad Eq_{10}=a-cW-W^3.$$

So in the direction $x=0$ we could have three invariant lines (which could coincide) and this occurs if and only if $d=0$. Thus we arrive at the family of systems

$$\dot{x}=a+cx+x^3,\quad \dot{y}=b+ex+lx^2+fy+2mxy \tag{4.3}$$

for which we calculate

$$H(X,Z)=Z(X^3+cXZ^2+aZ^3). \tag{4.4}$$

Considering systems (4.3) we calculate

$$\begin{aligned}
\mathcal{G}_1/H =& lX^4+X^3\big[4mY+2(e-lm)Z)\big]+X^2\big[(3f-4m^2)YZ+(3b-cl-lf-\\
&+-2em)Z^2\big]+X\big[-4fmYZ^2+(-2al-ef-2bm)Z^3\big]+(cf-f^2-\\
&-2am)YZ^3+(bc-ae-bf)Z^4\equiv F_1(X,Y,Z),\\
\mathcal{G}_2/H =&(X^3+cXZ^2+aZ^3)\Big[2lX^3+\big[X^2(6mY+(3e-2lm)Z\big]+X\big[(3f-\\
&-4m^2)YZ+(3b-cl-2em)Z^2\big]-2fmYZ^2+(-al-2bm)Z^3\Big]\equiv\\
&\equiv P^*(X,Z)F_2(X,Y,Z),\\
\mathcal{G}_3/H =&24(lX^2+2mXY+eXZ+fYZ+bZ^2)(X^3+cXZ^2+aZ^3)^2\equiv\\
&\equiv 24Q^*(X,Y,Z)\,[P^*(X,Z)]^2\,.
\end{aligned} \tag{4.5}$$

It is clear that systems (4.3) are degenerate if and only if the polynomials $P(x)$ and $Q(x,y)$ have a nonconstant common factor (depending on x) and this implies the existence of such a common factor (depending on X and Z) of the polynomials $P^*(X,Z)$ and $Q^*(X,Y,Z)$. So for non-degenerate systems the condition

$$R_X^{(0)}\left(P^*(X,Z),Q^*(X,Y,Z)\right)\neq 0 \tag{4.6}$$

must hold. We have the next lemma.

Lemma 4.2. *For a non-degenerate system* (4.3) *the polynomial* $P^*(X,Z)$ *could not be a factor of* $\mathcal{G}_1/H$*, i.e.* $P^*(X,Z)$ *does not divide* $F_1(X,Y,Z)$*.*

Proof. Suppose the contrary that $P^*(X,Z)$ divides $F_1(X,Y,Z)$. Then considering the form of the polynomial $P^*(X,Z)$ (which contains the term X^3) by Lemma 2.6 the following conditions are necessary and sufficient: $R_X^{(0)}(F_1,P^*)=R_X^{(1)}(F_1,P^*)=R_X^{(2)}(F_1,P^*)=0$. We calculate $R_X^{(2)}(F_1,P^*)=\big[(3f-4m^2)Y+(3b-2cl-lf-2em)\big]Z=0$ and this implies $f=4m^2/3$ and $b=2(3cl+3em+2lm^2)/9$. Then we obtain $R_X^{(1)}(F_1,P^*)=3^{-4}Z^4\big[12m(3c+4m^2)Y+(27al+18ce-6clm+24em^2+8lm^3)Z\big]^2=0$ and we consider two cases: $m\neq 0$ and $m=0$.

a) If $m\neq 0$ then we may assume $m=1$ and $e=0$ due to the change $(x,y,t)\to(mx,y-e/2m,t/m^2)$ and the above condition gives us $c=-4/3$ and $a=-16/27$. However in this case we have $R_X^{(0)}(P^*,Q^*)=0$, i.e. we get a contradiction with the condition (4.6).

b) Assume now $m=0$. Then we obtain

$$R_X^{(1)}(F_1,P^*)=(3al+2ce)^2Z^6=0,$$
$$R_X^{(0)}(F_1,P^*)=(27a^2+4c^3)\big[27a^2l^3+27ae(cl^2-e^2)+2c^2l(cl^2+9e^2)\big]Z^{12}/27=0,$$
$$R_X^{(0)}(P^*,Q^*)=\big[27a^2l^3+27ae(cl^2-e^2)+2c^2l(cl^2+9e^2)\big]/27\neq 0$$

and this implies $c\neq 0$, otherwise the second equality yields $a=0$ and then $R_X^{(0)}(P^*,Q^*)=0$. So $c\neq 0$ and the first equation gives $e=-3al/(2c)$ and then we arrive at the contradiction: $R_X^{(0)}(F_1,P^*)=\dfrac{l^3Z^{12}}{216c^3}(27a^2+4c^3)^3=0$, $R_X^{(0)}(P^*,Q^*)=\dfrac{l^3Z^6}{216c^3}(27a^2+4c^3)^2\neq 0$. This completes the proof of the lemma.

Now we examine the direction $y=0$. The following proposition holds.

Proposition 4.1. *For the existence of an invariant straight line of systems* (4.3) *in the direction* $y=0$ *it is necessary and sufficient*

$$l=0,\quad ef-2bm=0,\quad f^2+m^2\neq 0. \tag{4.7}$$

Proof. Indeed, considering the equations (2.17) for a system (4.3) we obtain $Eq_5=l$, $Eq_8=e-2mW$ and $Eq_{10}=b-fW$. Clearly, $Eq_5=0$ is equivalent to $l=0$. On the other hand in order to have an affine line in the direction $y=0$ the condition $f^2+m^2\neq 0$ is necessary.

Therefore the condition $\text{Res}_W(Eq_8, Eq_{10}) = ef - 2bm = 0$ is necessary and sufficient for the existence of a common solution $W = W_0$ of the equations $Eq_8 = 0$ and $Eq_{10} = 0$. This completes the proof of the proposition.

Now we find out coefficient criteria for a system (4.3) to belong to $\mathbb{CSL}_8$.

1) The case $m \neq 0,\ l \neq 0$. By Proposition 4.1 we could not have invariant line in the direction $y = 0$. So after the transformation $(x, y, t) \to (mx, -e/2m + ly, t/m^2)$ we can consider $l = m = 1$ and $e = 0$. As a result we arrive at the family of systems

$$\dot{x} = a + cx + x^3 \equiv P(x), \quad \dot{y} = \ b + x^2 + fy + 2xy \equiv Q(x, y). \tag{4.8}$$

Proposition 4.2. *Systems* (4.8) *possess invariant lines of total multiplicity 8 if and only if*

$$a = 0, \quad f = c = -\frac{4}{9}, \ b = \frac{4}{27}. \tag{4.9}$$

Proof. First we proof that the conditions (4.9) are sufficient for a system (4.8) to have invariant lines of total multiplicity eight.

Sufficiency. Assume that (4.9) are satisfied. Then for the corresponding system (4.8) we calculate $H(X, Y, Z) = -3^{-8}X^2(3X - 2Z)^3Z(3X + 2Z)$ and hence, we have 8 invariant straight lines (including the line at infinity).

Necessity. Consider systems (4.8) for which the polynomial H has the form (4.4). The degree of this polynomial equals four, but should be seven . Therefore we have to find out the conditions to increase the degree of the polynomial H up to seven, namely we have to find out additionally a common factor of degree three of the polynomials $\mathcal{G}_i,\ i = 1, 2, 3$ (see Lemma 2.4 and Notation 2.2).

Considering (4.5) for systems (4.8) we obtain $\mathcal{G}_1/H|_{Z=0} = X^3(X + 4Y)$. Therefore we conclude that all three polynomials could only have common factors of the form $X + \alpha = 0$, which by Remark **??** must be factors of the polynomial $P^*(X, Z)$. We observe that $P^*(X, Z)$ is a common factor of the polynomials $\mathcal{G}_2/H$ and $\mathcal{G}_3/H$ and, moreover, in the last one this factor is of the second degree.

According to Lemma 4.2 the polynomial $P^*(X, Z)$ could not be a factor of $\mathcal{G}_1/H$, i.e. of the polynomial $F_1(X, Y, Z)$. Thus not all the factors of the polynomial $P^*(X, Z)$ are also the factors in $F_1(X, Y, Z)$. This leads us to the conclusion that the polynomial $F_2(X, Y, Z)$ must have a common factor with $P^*(X, Z)$, i.e. the condition $R^{(0)}_X(F_2, P^*) = (8 + 27a + 18c)Z^3R^{(0)}_X(P^*, Q^*) = 0$ has to be fulfilled. Due to (4.6) this gives $c = -(8 + 27a)/18$ and we obtain that the polynomial $\psi = (3X - 2Z)$ is a common factor of the polynomials $F_2(X, Y, Z)$ and $P^*(X, Z)$. On the other hand it must be a factor in $F_1(X, Y, Z)$. We calculate

$$R^{(0)}_X(F_1, \psi) = -(8 + 27a + 18f)Z^3(12Y + 9fY + 4Z + 9bZ)/2 = 0,$$
$$R^{(0)}_X(P^*, Q^*) = (12Y + 9fY + 4Z + 9bZ)\Psi(Y, Z) \neq 0,$$

where $\Psi(Y,Z)$ is a polynomial. So the above conditions give us the equality $a = -2(4+9f)/27$ and then we obtain $f = c$. In this case calculations yield

$$\begin{aligned}\mathcal{G}_1/H =&\frac{1}{27}(3X-2Z)\big[9X^3+12X^2(3Y-Z)+3(9c-4)XYZ+\\ &+(27b-18c-8)XZ^2-2(4+9c)YZ^2\big]\equiv\frac{1}{27}(3X-2Z)F_1'(X,Y,Z),\\ \mathcal{G}_2/H =&\frac{1}{729}(3X-2Z)^2\big[18X^2+54XY-6XZ+27cYZ+(27b-9c-4)Z^2\big]\times\\ &\times(9X^2+6XZ+4Z^2+9cZ^2)\equiv\frac{1}{729}(3X-2Z)^2F_2'(X,Y,Z)\tilde{P}(X,Y,Z)\end{aligned}$$

and we obtain

$$\begin{aligned}R_X^{(0)}\left(F_1',\tilde{F}_2'\right) =&-729Z^2\big[36Y^2-3(4+9c)YZ+(4-27b+9c)Z^2\big]\Gamma(Y,Z),\\ R_X^{(0)}\left(F_1',\tilde{P}\right) =&729(4+9c)Z^4\Gamma(Y,Z),\\ R_X^{(0)}\left(P^*,Q^*\right) =&\frac{1}{729}Z^3(12Y+9cY+4Z+9bZ)\Gamma(Y,Z),\end{aligned}$$

where $\Gamma(Y,Z)$ is a polynomial. Since $R_X^{(0)}\left(F_1',\tilde{F}_2'\right)\neq 0$ due to $R_X^{(0)}(P^*,Q^*)\neq 0$, we deduce that for the existence of a common factor of degree 3 of the polynomials $\mathcal{G}_1/H$ and $\mathcal{G}_2/H$ the condition $R_X^{(0)}\left(F_1',\tilde{P}\right)=0$ is necessary, i.e. $c=-4/9$ and we get $c=f=-4/9$ and $a=0$. In this case we obtain

$$\begin{aligned}&\mathcal{G}_1/H=\frac{1}{9}X(3X-2Z)(3X^2+12XY-4XZ-8YZ+9bZ^2)\equiv\frac{1}{9}X(3X-2Z)F_1'',\\ &P^*(X,Z)=X(3X-2Z)(3X+2Z)/9\end{aligned}$$

and since X could not be a factor of $F_1''(X,Y,Z)$ and, moreover, as it was proved earlier the polynomial $P^*(X,Z)$ could not divide $\mathcal{G}_1/H$, we deduce that the factor of $F_1''(X,Y,Z)$ must be $3X-2Z$. So the condition $R_X^{(0)}(F_1'',3X-2Z)=3(27b-4)Z^2=0$ is necessary and this implies $b=4/27$, i.e. we arrive at the conditions (4.9) and this completes the proof of Proposition 4.2.

Considering the conditions (4.9) we obtain the family of systems which after the suitable transformation $(x,y,t)\to(2x/3,y+1/3,9t/4)$ becomes

$$\dot{x}=(x-1)x(1+x),\quad \dot{y}=x-y+x^2+3xy \tag{4.10}$$

with $H(X,Y,Z)=-X^2(X-Z)^3Z(X+Z)$. We observe that these systems possess 3 finite singularities: $(0,0)$, $(1,-1)$ and $(-1,0)$. On the other hand considering Lemma 2.1 for systems (4.10) we calculate: $\mu_0=\mu_1=\mu_2=\mu_3=\mu_4=\mu_5=0$, $\mu_6=8x^6\neq 0$. So by Lemma 2.1 all other 6 finite singular points have gone to infinity and collapsed with the singular point $[0,1,0]$ located on the “end" of the invariant line $x=0$.

Thus this system possesses 3 real distinct invariant affine lines (besides the double infinite line) and namely: one triple, one double and one simple, all real and distinct. Therefore we obtain the configuration *Config. 8.23* (Figure 4.1). We have proved the proposition.

2) The case $\boldsymbol{m \neq 0,\ l = 0}$. As it was mentioned earlier we may assume $m = 1$ and $e = 0$ due to the change $(x, y, t) \to (mx, y - e/2m, t/m^2)$. So we get the family of systems

$$\dot{x} = a + cx + x^3, \quad \dot{y} = b + fy + 2xy \tag{4.11}$$

which by Proposition 4.1 possess invariant line in the direction $y = 0$ if and only if $b = 0$.

a) *The subcase $b \neq 0$.* We claim that in this case the above systems could not have invariant lines of total multiplicity 8. Indeed, due to the rescaling $y \to by$ we can consider $b = 1$ and we obtain that for systems (4.11) the polynomial H of the form (4.4) has the degree 4, but should be 7. Moreover we have $\mathcal{G}_1/H|_{Z=0} = 4X^3Y$ and hence the polynomials $\mathcal{G}_k/H,\ k = 1, 2, 3$ (see their values (4.5) for $m = b = 1$ and $l = e = 0$) could have only the common factors of the form $X + \alpha Z$. Considering Remark **??** and Lemma 4.2 we arrive again at the conclusion that the polynomial $F_2(X, Y, Z)$ must have a common factor with $P^*(X, Z)$. We determine that for systems (4.11) $F_2(X, Y, Z) = (3X - 2Z)P^*(X, Z)Q^*(X, Y, Z)$ and hence due to the condition (4.6) and according to Lemma 4.2 (which says that $P^*(X, Z)$ could not divide $\mathcal{G}_1/H$) we conclude that $3X - 2Z$ must be a double factor in $\mathcal{G}_1/H$. However we obtain $R^{(1)}_X\left((3X - 2Z)^2, \mathcal{G}_1/H\right) = 162Z^3 \neq 0$, i.e. for systems (4.11) we could not increase the degree of $H(X, Y, Z)$ up to 7 and this completes the proof of our claim.

b) *The subcase $b = 0$.* We obtain the family of systems

$$\dot{x} = a + cx + x^3 \equiv P(x), \quad \dot{y} = y(f + 2x) \equiv y\tilde{Q}(x). \tag{4.12}$$

Proposition 4.3. *Systems* (4.12) *possess invariant lines of total multiplicity 8 if and only if one of the following sets of conditions holds:*

$$f = c,\ a = -\frac{2(4 + 9c)}{27},\ (4 + 3c)(4 + 9c) \neq 0; \tag{4.13}$$

$$f = \frac{-2(3c + 2)}{3},\ a = \frac{2(4 + 9c)}{27},\ (4 + 3c)(4 + 9c) \neq 0. \tag{4.14}$$

Proof . *Sufficiency.* Assume that (4.13) (respectively (4.14)) are satisfied. Then considering systems (4.12) we calculate $H(X, Y, Z) = 3^{-8}Y(3X - 2Z)^3Z(9X^2 + 6XZ + 4Z^2 + 9cZ^2)$ (respectively $H(X, Y, Z) = 3^{-9}2YZ(3X + 2Z)(9X^2 - 6XZ + 4Z^2 + 9cZ^2)^2$) and hence, we have 8 invariant straight lines, including the line at infinity. Moreover for the corresponding systems we calculate $R^{(0)}_X(\mathcal{G}_2/H, \mathcal{G}_1/H) = 3^{11}2(4 + 3c)^2(4 + 9c)Z^3$ (respectively $R^{(0)}_X(\mathcal{G}_2/H, \mathcal{G}_1/H) = -3^{15}(4 + 3c)^2(4 + 9c)Z^3$) and this leads to the following condition $(4 + 3c)(4 + 9c) \neq 0$ which does not allow us to have 9 invariant lines.

Necessity. For systems (4.12) we have $H(X, Y, Z) = YZ(X^3 + cXZ^2 + aZ^3)$. Thus according to Lemma 2.4 we conclude that we need additionally a non-constant factor of the second degree of H. For systems (4.12) we calculate (see Notation 2.2)

$$\mathcal{G}_1/H = 4X^3 - (4 - 3f)X^2Z - 4fXZ^2 - (2a - cf + f^2)Z^3,$$

$$\mathcal{G}_2/H = (3X - 2Z)(2X + fZ)(X^3 + cXZ^2 + aZ^3) \equiv (3X - 2Z)\tilde{Q}^*(X, Z)P^*(X, Z),$$

$$\mathcal{G}_3/H = 24(2X + fZ)(X^3 + cXZ^2 + aZ^3)^2 \equiv \tilde{Q}^*(X, Z)\left[P^*(X, Z)\right]^2.$$

We observe that $\mathcal{G}_1/H|_{Z=0} = 4X^3$ and we conclude that all three polynomials could not have Z as a common factor. On the other hand these polynomials do not depend on Y. So common factors of the above polynomials could be only factors of the form $X + \alpha Z$, which by Remark 3.4 must be also factors in $P^*(X,Z)$. So considering this remark and Lemma 4.2 we arrive at the two possibilities: the linear form $3X - 2Z$ either is a common factor of the polynomials $\mathcal{G}_1/H$ and $P^*(X,Z)$ or it is not.

i) Assume first that $3X - 2Z$ is a common factor of $\mathcal{G}_1/H$ and $P^*(X,Z)$. Then the following condition must be satisfied: $R_X^{(0)}(3X-2Z, P^*) = (8+27a+18c)Z^3 = 0$ and this implies $a = -2(4+9c)/27$. Herein we have $R_X^{(0)}(3X-2Z, \mathcal{G}_1/H,) = 9(c-f)(4+3f)Z^3 = 0$, $R_X^{(0)}(P^*(X,Z), Q^*(X,Z)) = (4+3f)(16+36c-12f+9f^2)Z^3/27 \neq 0$ and hence the condition $f = c$ must hold, which leads to the first two conditions (4.13).

ii) Suppose now that $3X - 2Z$ is not a common factor of $\mathcal{G}_1/H$ and $P^*(X,Z)$. Then clearly these polynomials must have a common factor of the second degree. So the conditions $R_X^{(0)}(P^*, \mathcal{G}_1/H) = (8a-4cf-f^3)\Phi_1(a,c,f)Z^9 = 0$, $R_X^{(1)}(P^*, \mathcal{G}_1/H) = \Phi_2(a,c,f)^4 = 0$ and $R_X^{(0)}(P^*, Q^*) = (4cf+f^3-8a)Z^3 \neq 0$ must hold, where $\Phi_1 = 8a + 27a^2 + 4c^2 + 4c^3 + 18af - f^3 - cf(4+3f)$, $\Phi_2 = 16c^2 + 2c(8+6f+3f^2) + 3(6af - 8a + 4f^2 + f^3)$. Due to $R_X^{(0)}(P^*, Q^*) \neq 0$ we must have $\Phi_1 = \Phi_2 = 0$ and we calculate $R_a^{(0)}(\Phi_1, \Phi_1) = 3(4+6c+3f)^2(4c+3f^2)(16+16c+3f^2) = 0$.

We claim that the condition $4+6c+3f = 0$ has to be satisfied for non-degenerate systems (4.12). Indeed assuming $c = -3f^2/4$ (respectively $c = -(16+3f^2)/16)$) we get that $4a + f^3$ (respectively $32a + 16f - f^3$) is a common factor of Φ_1 and Φ_2, however in this case the polynomial $R_X^{(0)}(P^*, Q^*)$ gives the value $-2(4a+f^3)Z^3 \neq 0$ (respectively $-(32a+16f-f^3)Z^3/4 \neq 0$).

So $4+6c+3f = 0$, i.e $f = -2(2+3c)/3$ and in this case the common factor of Φ_1 and Φ_2 is $(8-27a+18c)$. Hence the condition $\Phi_1 = \Phi_2 = 0$ implies $a = 2(4+9c)/27$ and this leads to the conditions (4.14). So the proposition is proven.

Next we construct the respective canonical forms of systems (4.12) when either the conditions (4.13) or (4.14) of Proposition 4.3 are satisfied.

α) Conditions (4.13). We observe that in this case due to a translation and an additional notation, namely $r = (4+3c)/3$, we arrive at the family of systems

$$\dot{x} = x(r + 2x + x^2), \quad \dot{y} = (r + 2x)y \tag{4.15}$$

for which we have $H(X,Y,Z) = X^3YZ(X^2+2XZ+rZ^2)$. So the polynomial $H(X,Y,Z)$ has the degree 7 and by Lemma 2.4 the above systems possess invariant lines of total multiplicity 8 (including the line at infinity, which is double). Now we need an additional condition under the parameter r which conserves the degree of the polynomial $H(X,Y,Z)$. For systems (4.15) we calculate $R_X^{(0)}(\mathcal{G}_3/H, \mathcal{G}_1/H) = 48r^3(8-9r)^2Z^5 \neq 0$. Consequently we get the condition $r(8-9r) \neq 0$ which for systems (4.12) is equivalent to $(4+3c)(4+9c) \neq 0$ (see the last condition from (4.13)).

Besides the infinite line $Z = 0$ (which is double) systems (4.15) possess six affine invariant lines, namely: $L_{1,2,3} = x,\ L_4 = y,\ L_{5,6} = r + 2x + x^2$.

We detect that the lines $L_{5,6} = 0$ are either complex or real distinct or real coinciding, depending on the sign of the discriminant of the polynomial $x^2 + 2x + r$, which equals $\Delta = 4(1-r)$. We also observe that systems (4.15) possess 3 finite singularities: $(0,0)$ and $(-1 \pm \sqrt{1-r}, 0)$ which are located on the invariant line $y = 0$. On the other hand considering Lemma 2.1 for systems (4.15) we calculate: $\mu_0 = \ldots = \mu_5 = 0, \quad \mu_6 = r^3x^6 \neq 0.$

So by Lemma 2.1 all other 6 finite singular points have gone to infinity and collapsed with the singular point $[0,1,0]$ located on the "end" of the invariant line $x = 0$. Moreover by this lemma systems (4.15) could be degenerate only if $r = 0$, and we observe that in this case the system indeed is degenerate.

We consider the three possibilities given by the value of the discriminant Δ.

- *The possibility* $\Delta > 0$. Then $1 - r > 0$, i.e. $r < 1$. We set the notation $1 - r = u^2$ (i.e. $r = 1 - u^2$) which leads to the systems

$$\dot{x} = (1 - u + x)x(1 + u + x), \quad \dot{y} = (1 - u^2 + 2x)y$$

possessing one triple and three simple distinct real invariant lines. Comparing the line $x = \pm u - 1$ with $x = 0$ we conclude that if $|u| > 1$ (i.e. $r < 0$) then in the direction $x = 0$ the triple invariant line is situated in the domain between two simple ones, whereas in the case $|u| < 1$ (i.e. $0 < r < 1$) the triple line is located outside this domain. As a result we get *Config. 8.24* in the case of $r < 0$ and *Config. 8.25* in the case of $0 < r < 1$.

- *The possibility* $\Delta = 0$. Then $r = 1$ and we obtain the configuration *Config. 8.26.*

- *The possibility* $\Delta < 0$. In this case $r > 1$ and we get systems possessing two complex, one simple and one triple real all distinct invariant lines and this leads to *Config. 8.27.*

β) Conditions (4.14). In this case after the translation of the origin of coordinates to the singular point $(-2/3, -e/2)$ and setting a new parameter $r = (4+3c)/3$ we obtain the systems

$$\dot{x} = (r - 2x + x^2)x, \quad \dot{y} = 2(x - r)y. \tag{4.16}$$

For these systems we have $H(X,Y,Z) = 2XYZ(X^2 - 2XZ + rZ^2)^2$. Besides the double infinite line systems (4.16) possess 4 affine invariant lines: $L_1 = x,\ L_2 = y,\ L_{3,4} = x^2 - 2x + r$, where the lines $L_{3,4} = 0$ are double ones. We denote by $\Delta = 4(1-r)$ the discriminant of the polynomial $x^2 - 2x + r$ and we observe that for $\Delta = 0$ (i.e. $r = 1$) the systems become degenerate.

We also observe that systems (4.16) possess 3 finite singularities: $(0,0)$ and $(1\pm\sqrt{1-r},0)$ which are located on the invariant line $y=0$. On the other hand considering Lemma 2.1 for systems (4.15) we calculate: $\mu_0=\ldots=\mu_5=0,\quad \mu_6=8(1-r)r^2x^6$. If $r(r-1)\neq 0$ by Lemma 2.1 all other 6 finite singular points have gone to infinity and collapsed with the singular point $[0,1,0]$ located on the "end" of the invariant line $x=0$. Moreover by this lemma systems (4.16) became degenerate only if either $r=0$ or $r=1$ and in both cases we get degenerate systems.

Thus we have the following two possibilities:

- *The possibility* $\Delta>0$. Then $r<1$ and denoting $r=1-v^2$ we obtain the systems

$$\dot{x}=(1+v-x)x(1-v-x),\quad \dot{y}=2(v^2-1+x)y \tag{4.17}$$

with $H(X,Y,Z)=2XYZ(X-Z-vZ)^2(X-Z+vZ)^2$. Examining the lines $x=1\pm v$ and $x=0$ we conclude that if $|v|>1$ then we get a simple invariant line between two double real lines in the directions $x=0$ and consequently we arrive at *Config. 8.28.* In the case of $|v|<1$ these two double real lines are located on the right–hand side of the simple invariant line. So we get *Config. 8.29.*

- *The possibility* $\Delta<0$. In this case $r>1$ and systems (4.16) possess 2 real simple, 2 complex double invariant lines, all distinct $\Rightarrow$ *Config. 8.30.*

3) The case $m=0,\ l\neq 0$. We claim that in this case systems (4.3) could not possess invariant lines of total multiplicity 8.

Indeed, since $l\neq 0$ by Proposition 4.1 we could not have a line in the direction $y=0$. Via the rescaling $(x\to x, y\to ly, t\to t)$ we can consider $l=1$ and therefore we arrive at the systems

$$\dot{x}=a+cx+x^3,\quad \dot{y}=\ b+ex+x^2+fy \tag{4.18}$$

for which the polynomial H has the form (4.4) and $\mathcal{G}_i/H$ $(i=1,2,3)$ are the polynomials (4.5) for the particular case $m=0$ and $l=1$. We observe that $\mathcal{G}_1/H|_{Z=0}=X^4$ and hence Z could not be a common factor of these polynomials. Since we have no invariant lines in the direction $y=0$, in what follows we shall examine only the conditions given by resultants with respect to X. According to (4.5) and condition (4.6) the polynomial $F_1(X,Y,Z)$ must have a common factor of degree 3 with $\left[P^*(X,Y)\right]^2$. For systems (4.18) we calculate Coefficient$[R_X^{(2)}\left(F_1,\left[P^*\right]^2\right),Y^4Z^4]=81f^4$. Clearly the condition $f=0$ is necessarily to get a common factor of degree 3. Then we have $R_X^{(0)}\left(F_1,\left[P^*\right]^2\right)=(27a^2+4c^3)^2[\Phi(a,b,c,e)]^2Z^{24}=0$ and $R_X^{(0)}(Q^*,P^*)=\Phi(a,b,c,e)Z^6\neq 0$ where $\Phi(a,b,c,e)$ is a polynomial. So the above conditions imply $27a^2+4c^3=0$. First we examine the possibility $a=0$ and we get $c=0$. Then we calculate $R_X^{(0)}(Q^*,P^*)=b^3Z^6\neq 0,\ R_X^{(2)}\left(F_1,\left[P^*\right]^2\right)=81b^4Z^8=0$ and we arrive at the contradictory condition $(0\neq b=0)$. So it remains to examine the case when $a\neq 0$. Since in this case $c\neq 0$ we denote $a=2a_1c$ and we obtain $c=-27a_1^2$. Then we calculate $R_X^{(0)}(Q^*, P^*) = (9a_1^2 + b - 3a_1e)^2(36a_1^2 + b + 6a_1e)Z^6 \neq 0$ and $R_X^{(1)}\left(F_1, [P^*]^2\right) = 2^3 3^{10} a_1^5 (9a_1^2 + b - 3a_1e)^3(36a_1^2 + b + 6a_1e)^2 Z^{15} = 0$ and we also get a contradiction which completes the proof of our claim.

4) The case $m = 0,\ l = 0$. We split our examination in two subcases: $e \neq 0$ and $e = 0$.

a) *The subcase $e \neq 0$.* Then due to the rescaling $(x, y, t) \to (ex, y, t/e^2)$ we can consider $e = 1$ and therefore we arrive at the systems

$$\dot{x} = a + cx + x^3, \quad \dot{y} = b + x + fy. \tag{4.19}$$

Proposition 4.4. *Systems* (4.19) *possess invariant lines of total multiplicity 8 if and only if the following conditions hold:*

$$f = -2c, \quad a = 0. \tag{4.20}$$

Proof. *Sufficiency.* Assume that (4.20) is satisfied. Then considering systems (4.19) we calculate $H(X, Y, Z) = XZ^2(X^2 + cZ^2)^2$ and hence, we have invariant straight lines of total multiplicity 8 (including the line at infinity). On the other hand we could not have 9 lines, because $R_X^{(0)}(G_2/H, G_1/H) = -27(2cY - bZ)^3 = 0$ if and only if $b = c = 0$. However in this case we get a degenerate system.

Necessity. For systems (4.19) we have $H(X, Y, Z) = Z^2(X^3 + cXZ^2 + aZ^3)$ and we observe that the degree of the polynomial H is 5. So we have to increase the degree of H up to 7. In other words we have to determine the conditions under which the three polynomials G_1/H, G_2/H and G_3/H have a common factor of degree 2. For these systems we calculate

$$\begin{aligned}
\mathcal{G}_1/H =& 2X^3 + 3fX^2Y + 3bX^2Z - fXZ^2 + f(c - f)YZ^2 + (bc - a - bf)Z^3,\\
\mathcal{G}_2/H =& 3X(X + fY + bZ)(X^3 + cXZ^2 + aZ^3) \equiv 3XQ^*P^*,\\
\mathcal{G}_3/H =& 24(X + fY + bZ)(X^3 + cXZ^2 + aZ^3)^2 \equiv 24Q^*[P^*]^2.
\end{aligned}$$

We observe that $G_1/H|_{Z=0} = 2X^3 + 3fX^2Y$ and hence Z could not be a common factor of these polynomials. For systems (4.19) we get $R_Y^{(0)}(G_3/H, G_1/H) = -24f(X^3 + cXZ^2 + aZ^3)^3$ which vanishes if and only if $f = 0$ and since $m = 0$, considering Proposition 4.1, we conclude that in this case we could not have a line in the direction $y = 0$. Thus all three mentioned polynomials could only have common factors of the form $X + \alpha = 0$, which by Remark **??** must be factors of the polynomial $P^*(X, Z)$. So considering this remark and Lemma 4.2 we arrive at the two possibilities: the linear form X either is not a common factor of the polynomials $\mathcal{G}_1/H = F_1(X, Y, Z)$ and $P^*(X, Z)$ (i.e. $a \neq 0$) or it is (i.e. $a = 0$).

i) Assume first that X is not a factor of $P^*(X, Z)$, i.e. we have to consider $a \neq 0$. According to (4.5) and condition (4.6) the polynomial $F_1(X, Y, Z)$ must have a common factor of degree 2 with $P^*(X, Y)$. Then considering systems (4.19) the following conditions must be satisfied:

$$R_X^{(0)}(F_1, P^*) = [27a^2 + (c - f)(2c + f)^2]Z^6\Psi(Y, Z) = 0,\ R_X^{(0)}(Q^*, P^*) = \Psi(Y, Z) \neq 0$$

where $\Psi(Y,Z)$ is a polynomial. So the condition $27a^2+(c-f)(2c+f)^2=0$ is necessary for the existence of a common factor of the polynomials F_1 and P^*. Then $(c-f)(2c+f)\neq 0$ (due to $a\neq 0$) and denoting $u=2c+f\neq 0$ (i.e. $f=u-2c$) we obtain $c=u/3-9a^2/u^2$ and $f=u-2c=(54a^2+u^3)/(3u^2)$. In this case we obtain

$$F_1=(uX+3aZ)F_1^*(X,Y,Z)/(3u^4),\;\; P^*=(uX+3aZ)(3uX^2-9aXZ+u^2Z^2)/(3u^2)$$

where $F_1^*(X,Y,Z)$ is a polynomial of the second degree. Assume first that $uX+3aZ$ is a factor in F_1^*. In this case it must be a factor in $3uX^2-9aXZ+u^2Z^2$ and therefore the following condition must hold: $R_X^{(0)}(uX+3aZ, 3uX^2-9aXZ+u^2Z^2)=u(54a^2+u^3)Z^2=0$. Since $u\neq 0$ we can set $a=a_1u$ and thus, we get $u=-54a_1^2$. Then $R_X^{(0)}(F_1^*, uX+3aZ)=18a_1(3a_1-b)Z^2=0$, $R_X^{(0)}(P^*,Q^*)=(b-3a_1)^2(6a_1+b)Z^3\neq 0$ and we arrive at the contradiction.

Now we consider that $uX+3aZ$ is not a factor in F_1^*. Then the polynomials F_1^* and $3uX^2-9aXZ+u^2Z^2$ must have a common factor, i.e. the following conditions hold: $R_X^{(0)}(F_1^*, 3uX^2-9aXZ+u^2Z^2)=27u^5Z^2F_1^{**}(Y,Z)=0$, $R_X^{(0)}(P^*,Q^*)=[(3a-bu)3uZ-(54a^2+u^3)Y]F_1^{**}(Y,Z)/(27u^6)\neq 0$ where $F_1^{**}(Y,Z)$ is a polynomial of the second degree. Since $c\neq 0$ in this case we also arrive at the contradictory condition.

ii) Assume now that X is a common factor of $P^*(X,Z)$, i.e. we have the condition $a=0$ which implies $\mathcal{G}_2/H=3X^2(X^2+cZ^2)Q^*$. Therefore either X^2 or X^2+cZ^2 must be a factor of F_1. In order to have X^2 as a common factor of the mentioned polynomial the condition $R_X^{(0)}(X^2,F_1)=R_X^{(1)}(X^2,F_1)=0$ must be satisfied. We calculate $R_X^{(1)}(X^2,F_1)=-fZ^2=0$, $R_X^{(0)}(X^2,F_1)=(c-f)^2Z^4(fY+bZ)^2=0$ and $R_X^{(0)}(P^*,Q^*)\mid_{\{c=f=0\}}=-b(b^2+c)Z^3$. It is evident that in order to have X^2 as a factor of the polynomial F_1 it is necessary the conditions $f=c=0$ and $b\neq 0$ to be satisfied, i.e. we get a particular case of the conditions (4.20). Since $b\neq 0$, due to the rescaling $\{x\to bx, y\to y/b, t\to t/b^2\}$ we can consider $b=1$. So we arrive at the system

$$\dot{x}=x^3,\quad \dot{y}=\ 1+x \tag{4.21}$$

for which $H(X,Z)=X^5Z^2$. This system possesses the affine invariant line of the multiplicity 5 in the direction $x=0$ and the infinite invariant line is of the multiplicity 3. Considering Lemma 2.1 for these systems we get $\mu_0=\ldots=\mu_8=0$, $\mu_9=9x^9\neq 0$. Therefore by Lemma 2.1 all 9 finite singular points have gone to infinity and collapsed with the singular point $[0,1,0]$ located on the "end" of the invariant line $x=0$. Consequently we get the configuration *Config. 8.33*.

Now we assume that X^2+cZ^2 is a factor of the polynomial F_1, i.e. the condition $R_X^{(0)}\big(X^2+cZ^2, F_1\mid_{\{a=0\}}\big)=R_X^{(1)}\big(X^2+cZ^2, F_1\mid_{\{a=0\}}\big)=0$ must hold. We calculate $R_X^{(1)}\big(X^2+cZ^2, F_1\mid_{\{a=0\}}\big)=-(2c+f)Z^2=0$ from which it results $f=-2c\neq 0$ and we obtain the conditions (4.20). Since $c\neq 0$ we may assume $b=0$ (applying the translation of the origin of coordinates at the point $x_0=0$, $y_0=b/(2c)$). Therefore we arrive at the non-degenerate systems depending on the parameter $c = \{-1, 1\}$ (applying a rescaling)

$$\dot{x} = x(c + x^2), \quad \dot{y} = x - 2cy. \tag{4.22}$$

For the above systems we have $H(X, Z) = XZ^2(X^2 + cZ^2)^2$. Thus beside the triple infinite invariant line systems (4.22) possess 5 invariant affine lines. More precisely, we have one simple and two double, all real and distinct if $c = -1$ and one simple real and two double complex if $c = 1$. On the other hand we observe that systems (4.22) possess 3 finite singularities: $(0, 0)$ and $\big(\pm\sqrt{-c}, \mp 1/(2\sqrt{-c})\big)$. Considering Lemma 2.1 for these systems we calculate: $\mu_0 = \ldots = \mu_5 = 0, \quad \mu_6 = -8c^3x^6 \neq 0$. Therefore by Lemma 2.1 all other 6 finite singular points have gone to infinity and collapsed with the singular point $[0, 1, 0]$ located on the "end" of the invariant line $x = 0$. Thus we get *Config. 8.31* if $c = -1$ and *Config. 8.32* if $c = 1$. The proposition is proven.

b) *The subcase* $e = 0$. Then we get the family of systems

$$\dot{x} = a + cx + x^3, \quad \dot{y} = b + fy \tag{4.23}$$

for which $H(a, X, YZ) = Z^2(fY + bZ)(X^3 + cXZ^2 + aZ^3)$. So the degree of H is six but should be seven. Therefore we need an additional common factor of $\mathcal{G}_i$, $i = 1, 2, 3$. We calculate $\mathcal{G}_1/H = 3X^2 + cZ^2 - fZ^2, \mathcal{G}_2/H = 3X(X^3 + cXZ^2 + aZ^3), \mathcal{G}_3/H = 24(X^3 + cXZ^2 + aZ^3)^2$ and we observe that these polynomials could not have as a common factor neither Z nor Y. So we examine their resultants with respect to X. We calculate

$$R_X^{(0)}(\mathcal{G}_1/H, P^*) = [27a^2 + (c - f)(2c + f)^2]Z^6 = 0, \; R_X^{(0)}(P^*, Q^*) = (fY + bZ)^3 \neq 0,$$

which implies $27a^2 + (c - f)(2c + f)^2 = 0$. We observe that $(c - f)(2c + f) \neq 0$, otherwise we get $a = 0$ and this leads to systems with invariant lines of total multiplicity 9.

Denoting $u = 2c + f \neq 0$ (i.e. $f = u - 2c$) we obtain $c = u/3 - 9a^2/u^2$ and $f = u - 2c = (54a^2 + u^3)/(3u^2)$. So we get the family of systems

$$\dot{x} = \frac{1}{3u^2}(3a + ux)(u^2 - 9ax + 3ux^2), \quad \dot{y} = b + \frac{54a^2 + u^3}{3u^2}y. \tag{4.24}$$

Without loss of generality we may assume $b \neq 0$, because in the case $b = 0$ we must have $54a^2 + u^3 \neq 0$ (otherwise we get degenerate systems) and then via a translation $y \to y + y_0$ (with $y_0 \neq 0$) we obtain $b \neq 0$. So applying the translation of the origin of coordinates at the point $(-3a/u, 0)$, after the suitable rescaling $\{x \to -(9ax)/u, y \to bu^2y/(81a^2), t \to tu^2/(81a^2)\}$ systems (4.24) become

$$\dot{x} = rx + x^2 + x^3, \quad \dot{y} = 1 + ry, \tag{4.25}$$

where $r = (54a^2 + u^3)/(243a^2)$. For these systems we calculate $H = X^2(rY + Z)Z^2(X^2 + XZ + rZ^2)$ and $R_X^{(0)}(\mathcal{G}_2/H, \mathcal{G}_1/H) = 3(9r - 2)Z^3 \neq 0$ and this leads to the condition

$9r - 2 \neq 0$ which guarantee the non-existence of nine invariant lines. We observe that the infinite invariant line Z=0 is triple if $r \neq 0$ and it has multiplicity four in the case $r = 0$.

i) *The possibility* $r \neq 0$. In this case the geometry of the configuration depends on the sign of the discriminant Δ of the polynomial $x^2 + x + r$, i.e. $\Delta = 1 - 4r$. Accordingly we conclude that besides the triple infinite invariant line the systems (4.24) possess 5 affine lines which are as follows:

$$\begin{array}{lll} \Delta > 0 \ \ (\text{i.e. } 0 \neq r < 1/4) & \Rightarrow & \text{3 simple, 1 double, all real and distinct,} \\ \Delta = 0 \ \ (\text{i.e. } r = 1/4) & \Rightarrow & \text{1 simple, 2 double, all real and distinct,} \\ \Delta < 0 \ \ (\text{i.e. } r > 1/4) & \Rightarrow & \text{2 real simple, 1 complex double.} \end{array}$$

On the other hand considering Lemma 2.1 we calculate: $\mu_0 = \ldots = \mu_5 = 0$, $\mu_6 = r^3x^6$, $\mu_7 = r^2x^6(3x - ry)$, $\mu_8 = rx^6(3x^2 - 2rxy + r^3y^2)$, $\mu_9 = 9x^7(x^2 - rxy + r^3y^2)$. Since $r \neq 0$ by Lemma 2.1 only 6 finite singular points have gone to infinity and collapsed with the singular point $[0, 1, 0]$ located on the "end" of the invariant line $x = 0$. Other three finite points are $(0, -1/r)$ and $\big((-1 \pm \sqrt{1 - 4r})/2, -1/r\big)$ (located on the invariant line $ry + 1 = 0$).

Moreover, in the case of $\Delta > 0$, denoting $1 - 4r = v^2$ (i.e. $r = (1 - v^2)/4$) we obtain the systems

$$\dot{x} =(1 - v + 2x)x(1 + v + 2x)/4, \quad \dot{y} = \ 1 + (1 - v^2)y/4.$$

We compare the lines $x = (-1 \pm v)/2$ with $x = 0$ and conclude that if $|v| > 1$, i.e. $r < 0$ (respectively $0 < |v| < 1/4$, i.e. $0 < r < 1/4$) then the double real invariant line is located (respectively is not located) between two simple ones and we arrive at the configuration *Config. 8.34* (respectively *Config. 8.35.*).

Additionally, we have the configuration *Config. 8.36* in the case of $\Delta = 0$ (i.e. $r = 1/4$) and *Config. 8.37* in the case of $\Delta < 0$ (i.e. $r > 1/4$).

ii) *The possibility* $r = 0$. In this case we get the system

$$\dot{x} = x^2(x + 1), \quad \dot{y} = 1 \tag{4.26}$$

with $H(X, Z) = X^3Z^3(X + Z)$. Therefore besides the infinite line of the multiplicity four this system possesses 2 distinct affine invariant lines (one of the multiplicity 3 and one simple), and namely: $L_{1,2,3} = x$, $L_4 = x + 1$. Since in this case we obtain $\mu_i = 0$ $(i = 0, 1, \ldots, 8)$ and $\mu_9 = 9x^9 \neq 0$, by Lemma 2.1 all 9 finite singular points have gone to infinity and collapsed with the the same singular point $[0, 1, 0]$. As a result we get the configuration *Config. 8.38.* Thus considering the above results we arrive at the following proposition.

Proposition 4.5. *The systems* (4.23) *possess invariant lines of total multiplicity eight if and only if*

$$27a^2 + (c - f)(2c + f)^2 = 0, \quad a \neq 0. \tag{4.27}$$

4.1.2. Systems with cubic homogeneous parts $(x^3, 3x^2y)$

Considering the homogeneities $(x^3, 3x^2y)$ due to a translation we may assume $g = l = 0$ for cubic systems (4.1) and we arrive at the family of systems

$$\dot{x} = a + cx + dy + 2hxy + ky^2 + x^3, \quad \dot{y} = b + ex + fy + 2mxy + ny^2 + 3x^2y \tag{4.28}$$

with $C_3 = -2x^3y$. Taking into account Remark 3.1 for these systems we calculate $\mathcal{L}_1 = 20736x^2\big[(2h+n)x+ky\big]$ and thus, the condition $\mathcal{L}_1 = 0$ is equivalent to $n = -2h$ and $k = 0$. Then we have $\mathcal{K}_1 = 0$ and $\mathcal{L}_2 = 20736x[(21d - 8hm) + 48h^2y] = 0$ which implies $h = d = 0$. So for $n = k = h = d = 0$ we get the following family of systems:

$$\dot{x} = a + cx + x^3 \equiv P(x), \quad \dot{y} = b + ex + fy + 2mxy + 3x^2y \equiv Q(x, y) \tag{4.29}$$

for which we calculate

$$\begin{aligned} H(X, Y, Z) = X^3 + cXZ^2 + aZ^3, \quad \mathcal{G}_1/H = F_1(X, Y, Z), \\ \mathcal{G}_2/H = F_2(X, Y, Z)(X^3 + cXZ^2 + aZ^3) = F_2(X, Y, Z)P^*(X, Z), \\ \mathcal{G}_3/H = -48Q^*(X, Y, Z)[P^*(X, Z)]^2, \end{aligned} \tag{4.30}$$

where $F_1(X, Y, Z)$, $F_2(X, Y, Z)$ are homogeneous polynomials in X, Y and Z of the degree five and four, respectively. It is clear that these systems are degenerate if and only if the polynomials $P(x)$ and $Q(x, y)$ have a non-constant common factor (depending on x), i.e. the following condition must hold:

$$\Phi(y) \equiv R_x^{(0)}(P(x), Q(x, y)) \neq 0. \tag{4.31}$$

Remark 4.2. *Systems* (4.29) *could not have more then one invariant line in the direction* $y = 0$. *This follows directly form Remark 3.4 and the fact that* $Q(x, y)$ *in these systems has degree one with respect to* y.

Systems (4.29) possess invariant lines of total multiplicity 4, including the infinite one , but we need 8 invariant lines (considered with their multiplicities), i.e. additionally we have to obtain a common factor of fourth degree of the polynomials $\mathcal{G}_i/H$, $i = 1, 2, 3$. In order to reach this situation we examine the directions $x = 0$ and $y = 0$.

Since in the direction $x = 0$ we already have 3 invariant lines $x^3 + cx + a = 0$ (which could coincide), we consider the equations (2.17) only for the direction $y = 0$. Considering systems (4.29) and Remark 2.1 we have $Eq_5 = -3W$, $Eq_8 = e - 2mW$ and $Eq_{10} = b - fW$. Evidently that these equations could have only one common solution ($W_0 = 0$) and for this it is necessary and sufficient $e = b = 0$. So in what follows we examine two cases: $e^2 + b^2 = 0$ and $e^2 + b^2 \neq 0$.

1) The case $e^2 + b^2 = 0$. Then we get the family of systems

$$\dot{x} = a + cx + x^3, \quad \dot{y} = (f + 2mx + 3x^2)y \equiv \tilde{Q}(x)y \tag{4.32}$$

for which we calculate

$$H(X,Y,Z) = Y(X^3 + cXZ^2 + aZ^3),\ \mathcal{G}_1/H = \tilde{F}_1(X,Z),$$
$$\mathcal{G}_2/H = \tilde{F}_2(X,Z)P^*(X,Z),\ \mathcal{G}_3/H = -48\tilde{Q}^*(X,Z)[P^*(X,Z)]^2.$$

The polynomial H is of degree 4 and thus we need to increase its degree up to 7. In order to reach this situation we have to obtain a common factor of degree 3 of the above polynomials. By Remark 4.2 systems (4.32) have only one invariant line in the direction $y = 0$. Moreover since $\mathcal{G}_1/H\ |_{(Z=0)}= -6X^4$ we also could not have Z as a common factor. Therefore all three polynomials could have only factors of the form $X + \alpha$, which must be factors of the polynomial $P^*(X,Z)$ (see Remark 3.4).

Thus, in order to get a common factor of the third degree of the mentioned polynomials, the following condition must hold: $R^{(0)}_X([P^*]^2, \tilde{F}_1) = R^{(1)}_X([P^*]^2, \tilde{F}_1) = R^{(2)}_X([P^*]^2, \tilde{F}_1) = 0$.

Considering systems (4.32) we calculate $R^{(2)}_X([P^*]^2, \tilde{F}_1) = [\Psi(a,c,f,m)]^2 Z^8 = 0$ where $\Psi(a,c,f,m) = 3(c-f)(3c-f) - 36am + 8(5c-3f)m^2 + 16m^4$.

a) Assume first $m = 0$. Then the last condition is equivalent to $(3c-f)(c-f) = 0$, i.e. we need to examine two cases: $f = 3c$ and $f = c$.

Assuming $f = 3c$ we calculate $R^{(0)}_X([P^*]^2, \tilde{F}_1) = 46656a^4c^6Y^6Z^{24} = 0$ and and considering (4.31) we have $\Phi = R^{(0)}_x(P(x), \tilde{Q}(x)) = 27a^2 \neq 0$. Thus we arrive at the condition $c = 0$ which implies $f = 0$ and so we obtain the systems

$$\dot{x} = a + x^3, \quad \dot{y} = 3x^2y. \tag{4.33}$$

We remark that this family of systems is a subfamily of (4.34) bellow and we will examine it together with the family of systems (4.34).

Now we consider $f = c \neq 0$ which implies $\Phi = (27a^2 + 4c^3) \neq 0$. In this case we obtain $R^{(0)}_X([P^*]^2, \tilde{F}_1) = R^{(1)}_X([P^*]^2, \tilde{F}_1) = R^{(2)}_X([P^*]^2, \tilde{F}_1) = 0$ and $R^{(3)}_X([P^*]^2, \tilde{F}_1) = -216aZ^3 \neq 0$ (i.e. the condition $a \neq 0$ is necessary, otherwise we get invariant lines of the total multiplicity 9). As a result we arrive at the family of systems

$$\dot{x} = a + cx + x^3, \quad \dot{y} = (c + 3x^2)y \tag{4.34}$$

for which $H(X,Y,Z) = 3Y(X^3 + cXZ^2 + aZ^3)^2$, i.e. these systems possess invariant line of total multiplicity 8.

We observe that systems (4.33) belong to this family for $c = 0$. So we allow the parameter c to be zero in order to include (4.33) in (4.34). It is clear that the polynomial $a + cx + x^3$ has at least one real solution, say x_0. Therefore due to the translation of the origin of coordinates to the singular point $(x_0, 0)$ systems (4.34) become of the form

$$\dot{x} = x(e + gx + x^2), \quad \dot{y} = (e + 2gx + 3x^2)y \tag{4.35}$$

where $e = c+3x_0^2$ and $g = 3x_0$ and we calculate $H(X,Y,Z) = X^2Y(X^2+gXZ+eZ^2)^2$. On the other hand considering systems (4.35) we calculate $R_X^{(0)}(\mathcal{G}_2/H, \mathcal{G}_1/H) = 24g(9e-2g^2)Z^3$ and therefore the condition $g(9e-2g^2) \neq 0$ guaranties us to have no 9 invariant lines considered with their multiplicities. Taking into account that $g \neq 0$ we can set $g = 1$ due to the rescaling $(x,y,t) \mapsto (gx, y, t/g^2)$ and we obtain

$$\dot{x} = x(r + x + x^2), \quad \dot{y} = (r + 2x + 3x^2)y. \tag{4.36}$$

We also observe that systems (4.36) possess 3 finite singularities: $(0,0)$ and $\left(\frac{-1\pm\sqrt{1-4r}}{2}, 0\right)$ which are located on the invariant line $y = 0$. On the other hand considering Lemma 2.1 for these systems we calculate: $\mu_0 = \ldots = \mu_5 = 0, \quad \mu_6 = r^2(4r-1)x^6$. If $r(4r-1) \neq 0$ by Lemma 2.1 all other 6 finite singular points have gone to infinity and collapsed with the singular point $[0,1,0]$ located on the “end" of the invariant line $x = 0$. Moreover by Lemma 2.1 systems (4.36) become degenerate only if either $r = 0$ or $r = 1/4$ and in both cases we indeed get degenerate systems.

Thus, systems (4.35) possess 7 affine invariant lines and the type of some of these lines depends on the polynomial $1-4r$ =Discriminant$[x^2 + x + r, x]$. Therefore we have the following two possibilities:

- *The possibility* $1-4r > 0$. Then we denote $1-4r = u^2 \neq 0$ $\big($i.e. $r = (1-u^2)/4 \neq 0\big)$ and considering (4.35) we get the systems

  $$\dot{x} = x(1+2x-u)(1+2x+u)/4, \quad \dot{y} = (1-u^2+8x+12x^2)y/4 \tag{4.37}$$

  with $H(X,Y,Z) = 2^{-6}X^2Y(2X+Z-uZ)^2(2X+Z+uZ)^2$. So in this case we obtain 1 simple and 3 double invariant straight lines, all real and distinct. Evidently that the condition $r(4r-1) \neq 0$ is equivalent to $u(1-u^2) \neq 0$.

- *The possibility* $1-4r < 0$. Then denoting $1-4r = -u^2 \neq 0$ (i.e. $r = (1+u^2)/4 \neq 0$) we arrive at the systems

  $$\dot{x} = x[(2x+1)^2 + u^2]/4, \quad \dot{y} = (1+u^2+8x+12x^2)y/4 \tag{4.38}$$

  for which we have $H(X,Y,Z) = 2^{-6}X^2Y(4X^2+4XZ+Z^2+u^2Z^2)^2$. Clearly, in this case we obtain the following types of invariant straight lines: one double real, two double complex and one simple real, all distinct.

More exactly, systems (4.36) possess the configuration *Config. 8.39* if $1-4r > 0$ and *Config. 8.40* in the case of $1-4r < 0$ (see Figure 4.1).

b) Now we assume $m \neq 0$. We may set $m = 1$ $\big($due to the rescaling $(x,y,t) \mapsto (mx, y, t/m^2)\big)$ and considering systems (4.32) we calculate

$$R_X^{(0)}([P^*]^2, \tilde{F}_1) = [\Psi_1(a,c,f)]^2[\Psi_2(a,c,f)]^2Z^{24} = 0, \quad \Phi(a,c,f) = \Psi_2(a,c,f) \neq 0,$$
$$\Psi_1(a,c,f) = 8a + (c-f)[c(4+c) - 2cf + f^2].$$

Clearly the condition $\Psi_1(a,c,f)=0$ is necessary and sufficient to have a common factor of $[P^*]^2$ and $\tilde{F}_1$ for non-degenerate systems (4.32). Then $a=(c-f)\big(c(4+c)-2cf+f^2\big)/8$ and in this case we have

$$R_X^{(1)}([P^*]^2,\tilde{F}_1)=(4+3c-3f)(4c+3c^2-6cf+3f^2)^3(16+16c+3c^2-6cf+3f^2)^2Z^{15}/64,$$
$$\Phi=(4c+3c^2-6cf+3f^2)^2(16+16c+3c^2-6cf+3f^2)/64\neq 0.$$

So the equality $R_X^{(1)}([P^*]^2,\tilde{F}_1)=0$ implies $f=(4+3c)/3$ and then $R_X^{(2)}([P^*]^2,\tilde{F}_1)=256(4+3c)^2Z^8/9=0$ which contradicts $\Phi=4(4+3c)^3/27\neq 0$. So in the case $m\neq 0$ systems (4.32) could not have invariant lines of multiplicity 8.

2) The case $e^2+b^2\neq 0$. We again consider systems (4.29) which already possess 3 lines in the direction $x=0$. Taking into consideration that we are in the case of non-existence of an invariant line in the direction $y=0$, in order to increase the degree of the polynomial H we need a common factor of the degree 4 of the polynomials $\mathcal{G}_i/H, i=1,2,3$. By Lemma 2.6 this happens if and only if it holds $R_X^{(0)}([P^*]^2,F_1)=R_X^{(1)}([P^*]^2,F_1)=R_X^{(2)}([P^*]^2,F_1)=R_X^{(3)}([P^*]^2,F_1)=0$. We calculate $R_X^{(3)}([P^*]^2,F_1)=-8[27a-2m(18c-9f+8m^2)]Y^3Z^3+12(3ce+6bm-4em^2)Y^2Z^4+e^3Z^6=0$ and this condition is equivalent to $e=bm=27a-2m(18c-9f+8m^2)=0$. Since $e^2+b^2\neq 0$ we obtain $e=m=a=0$ and $b\neq 0$. In this case we calculate $R_X^{(0)}([P^*]^2,F_1)=(c-f)^6Z^{24}(3cY-fY-bZ)^4(fY+bZ)^2=0$ and since $b\neq 0$ it results $f=c$. Consequently we obtain the family of systems $\dot{x}=x(c+x^2),\ \dot{y}=b+cy+3x^2y$ with $H(X,Y,Z)=3X^3(X^2+cZ^2)^2$ and after the rescaling $(x,\ y)\to(x,\ by)$ we arrive at the one-parameter family of systems

$$\dot{x}=x(c+x^2),\quad \dot{y}=1+cy+3x^2y. \tag{4.39}$$

Here we may assume $c=\{-1,0,1\}$ due to the rescaling $(x,y,t)\mapsto(|c|^{1/2}x,|c|^{-1}y,|c|^{-1}t)$.

According to Lemma 2.1 for these systems we calculate: $\mu_0=\mu_1=\mu_2=\mu_3=\mu_4=\mu_5=0$, $\mu_6=4c^3x^6$. If $c\neq 0$ systems (4.39) possess 3 finite singularities: $(0,-1/c)$ and $\big(\pm\sqrt{-c},1/(2c)\big)$ and by Lemma 2.1 all other 6 finite singular points have gone to infinity and collapsed with the singular point $[0,1,0]$ located on the "end" of the invariant line $x=0$.

If $c=0$ then $\mu_6=\mu_7=\mu_8=0$ and $\mu_9=9x^9$. So the system is non-degenerate and all 9 finite singularities have gone to infinity and collapsed with the same singular point.

Thus in the case $c\neq 0$ a system (4.39) possesses three distinct invariant affine lines (one triple and two doubles), and namely: $L_{1,2,3}=x,\ L_{4,5}=x-\sqrt{-c},\ L_{6,7}=x+\sqrt{-c}$.

Moreover for $c<0$ we have real invariant straight lines whereas for $c>0$ we get two complex invariant lines. As a result we obtain the configuration given by *Config. 8.41* in the case $c=-1$ and by *Config. 8.43* in the case $c=1$.

If $c=0$ then the invariant affine line $x=0$ becomes of multiplicity 7 and we arrive at the configuration given by *Config. 8.42*.

The above results lead as to the following proposition:

Proposition 4.6. *Systems* (4.29) *possess invariant lines of total multiplicity eight if and only if the following set of conditions holds:*

$$e = m = f - c = 0 \quad \text{and either: } b = 0 \text{ and } a(27a^2+4c^3) \neq 0 \quad \text{or} \quad b \neq 0 \text{ and } a = 0. \quad (4.40)$$

4.1.3. Systems with cubic homogeneous parts $\big((u+1)x^3, ux^2y\big)$

In this case, considering (4.1) and the cubic homogeneity $\big((u+1)x^3, ux^2y\big)$ via a translation of the origin of coordinates we can consider $l = m = 0$ (since $u \neq 0$) and therefore we get the cubic systems

$$\dot{x} = a + cx + dy + gx^2 + 2hxy + ky^2 + (1+u)x^3, \quad \dot{y} = b + ex + fy + ny^2 + ux^2y. \quad (4.41)$$

Now we force the necessary conditions $\mathcal{K}_4 = \mathcal{K}_5 = \mathcal{K}_6 = 0$ (see Remark 3.1) which correspond to the type of configuration (3, 2, 1, 1) to be satisfied. We calculate

$$\text{Coefficient}[\mathcal{K}_4, x^2y] = ku(3-u)/9 = 0, \;\; \text{Coefficient}[\mathcal{K}_5, xy^3] = -40k^2(6+u)(3+4u)/3 = 0$$

which leads to $k = 0$ and then we obtain $\mathcal{K}_4 = -u(3n + 2hu + nu)x^3/9 = 0$. Since $u \neq 0$ the last condition is equivalent to $h = -n(3+u)/(2u)$ which implies Coefficient$[\mathcal{K}_5, x^3y] = 10n^2(u^3 - 54 - 90u - 33u^2)/u^2$, Coefficient$[\mathcal{K}_6, x^8y^3] = 2n^3(13150u^6 - 125874 - 492669u - 774792u^2 – 638868u^3 – 268688u^4 – 23699u^5)/(9u^2)$. It is easy to check that Coefficient$[\mathcal{K}_5, x^3y] =$ Coefficient$[\mathcal{K}_6, x^8y^3] = 0$ if and only if $n = 0$ (which imply $h = 0$) and in this case we get $\mathcal{K}_5 = 10du^2 = 0$ which gives $d = 0$. In this case it remains to examine only the condition $\mathcal{K}_6 = 0$.

Thus taking into consideration the conditions $k = h = n = d = 0$ systems (4.41) become

$$\dot{x} = a + cx + gx^2 + (u+1)x^3, \quad \dot{y} = b + ex + fy + ux^2y \quad (4.42)$$

for which we calculate $\mathcal{K}_6 = 40u^4(81fg - 369au + 68cgu + 136fgu - 96au^2)x^{11}/27 = 0$ and $\mathcal{V}_1\mathcal{V}_3 = -512u^2(3+u)(3+2u)x^8 \neq 0$ (see Lemma 4.1), i.e. for these systems the following conditions hold:

$$\psi \equiv 81fg - 369au + 68cgu + 136fgu - 96au^2 = 0 = \mathcal{K}_6, \;\; u(u+3)(2u+3) \neq 0. \quad (4.43)$$

In addition, considering (2.17) and Remark 2.1 for systems (4.41) we examine the direction $y = 0$: $Eq_5 = -uW$, $Eq_8 = e$, $Eq_{10} = b - fW$. We see that the equations Eq_8 and Eq_{10} could have only one common solution ($W_0 = 0$) and for this it is necessary and sufficient $b = e = 0$. So need to examine two cases: $e^2 + b^2 = 0$ and $e^2 + b^2 \neq 0$. In [17] we have proved that in the case of $e^2 + b^2 \neq 0$ we could not have systems which belong to $\mathbb{CSL}_8$.

Here we consider only the case $e^2 + b^2 = 0$, i.e. when there exists an invariant line in the direction $y = 0$. In this case we get the systems

$$\dot{x} = a + cx + gx^2 + (u+1)x^3 \equiv P(x), \quad \dot{y} = y(f + ux^2) \equiv y\tilde{Q}(x) \quad (4.44)$$

for which the conditions (4.43) hold. We consider two possibilities: $u+1 \neq 0$ and $u+1=0$.

1) The subcase $u+1 \neq 0$ Then for the above systems we calculate

$$H(X,Y,Z) = Y[X^3(1+u) + gX^2Z + cXZ^2 + aZ^3] \equiv XP^*(X,Z),$$
$$\mathcal{G}_1/H = F_1(X,Z), \;\; \mathcal{G}_2/H = P^*(X,Z)F_2(X,Z), \;\; \mathcal{G}_3/H = 24\tilde{Q}^*(X,Z)\big[P^*(X,Z))\big]^2.$$

Systems (4.44) are non-degenerate if and only if $\Phi \equiv R_X^{(0)}\big(P(x), \tilde{Q}(x)\big) \neq 0$. By Remark 4.2 we could not have invariant lines in the direction $y=0$ except the existent invariant line $y=0$. Moreover due to $\mathcal{G}_1/H\,|_{(Z=0)} = uX^4 \neq 0$ (as $u \neq 0$) we obtain that Z could not be a common factor of these polynomials. Therefore the degree of the polynomial $H(x,y)$ could be increased up to seven only with the factors of the form $X + \alpha Z$. Considering Remark 3.4 we deduce that these factors must be factors of the polynomial $P^*(X,Z)$. So $F_1(X,Z)$ must have a common factor of degree 3 with $\big[P^*(X,Z)\big]^2$. Therefore by Lemma 2.6 the conditions $R_X^{(0)}\big(F_1, [P^*(X,Z)]^2\big) = R_X^{(1)}\big(F_1, [P^*(X,Z)]^2\big) = R_X^{(2)}\big(F_1, [P^*(X,Z)]^2\big) = 0$ and $R_X^{(3)}\big(F_1, [P^*(X,Z)]^2\big) \neq 0$ must hold. We calculate

$$R_X^{(0)}\big(F_1, [P^*]^2\big) = [\Phi]^2\Gamma_1^2 Z^{24}, \;\; R_X^{(1)}\big(F_1, [P^*]^2\big) = -2\Phi\Gamma_2 Z^{15}, \;\; R_X^{(2)}\big(F_1, [P^*]^2\big) = \Gamma_3 Z^8,$$

where $\Gamma_1(a,c,f,g,u)$, $\Gamma_2(a,c,f,g,u)$ and $\Gamma_3(a,c,f,g,u)$ are some polynomials of total degree 5, 11 and 12, respectively. Evidently since $\Phi \neq 0$ from the above conditions it results $\Gamma_1 = \Gamma_2 = \Gamma_3 = 0$.

We claim that for non-degenerate systems (4.44) the polynomials Γ_1 and Γ_2 vanish if and only if they have a common factor depending on the parameter a. Indeed, we observe that Γ_1 is of degree two with respect to a and moreover Coefficient$[\Gamma_1, a^2] = (3+2u)^3 \neq 0$. This means that the solution of the equation $\Gamma_1 = 0$ must depend on a. On the other hand the degree of Γ_2 with respect to a equals 3 and Coefficient$[\Gamma_2, a^3] = u^2(u+1)(3+2u)(u+2) = 0$ if and only if $u=-2$. In this case we obtain

$$\Gamma_2 = (f-2g^2)[8a^2g + a(3f^2 - 2cf - 12cg^2 + 14fg^2 - 8g^4) + g(2c^2f - 7cf^2 +$$
$$+4f^3 + 4c^2g^2 - 2cfg^2 + 2f^2g^2 - 4fg^4)]$$

and hence for $u=-2$ and $f=2g^2$ we have $\Gamma_2 = 0$ and this solution does not depend of a. However in this case the condition $\Gamma_1 = -(a - cg + 2g^3)^2 = 0$ gives $a = g(c-2g^2)$ and this leads to degenerate systems.

If $(f-2g^2) \neq 0$ then in order to impose the polynomial Γ_2 to vanish for any value of the parameter a it is necessary $g=0$ and then we calculate

$$\Gamma_1 = cf^2 - f^3 - a^2, \;\; \Gamma_2 = -a(2c-3f)f^2, \;\; \Phi = 8(cf^2 - f^3 - a^2) + (2c-3f)^2 f.$$

Clearly the conditions $(2c-3f)f = 0$ and $\Gamma_1 = 0$ implies $\Phi = 0$, i.e. systems become degenerate and this completes the proof of our claim.

Thus the polynomials Γ_1 and Γ_2 must have a common factor depending on a and by Lemma 2.5 the condition $R_a^{(0)}(\Gamma_1,\Gamma_2)=0$ must hold. We calculate

$$R_a^{(0)}(\Gamma_1,\Gamma_2)=(3+2u)^4\Upsilon_1(c,f,g,u)[\Upsilon_2(c,f,g,u)]^4\Upsilon_3(c,f,g,u)\Upsilon_4(c,f,g,u)=0$$

where $\Upsilon_1(c,f,g,u)=f(3+u)^2-9c+3g^2-6cu-cu^2$, $\Upsilon_2(c,f,g,u)=f(1+u)(3+2u)+6c-2g^2+7cu+2cu^2$, $\Upsilon_3(c,f,g,u)=4fg^2u+(3f-cu+3fu)^2$ and $\Upsilon_4(c,f,g,u)=u^2[g^2-c(2+u)^2]^2+f^2(1+u)^2[3+u(3+u)]^2-2fu(1+u)[c(2+u)^2\big(3+u(3+u)\big)-g^2\big(5+u(5+u)\big)]$.

i) Assume $\Upsilon_1(c,f,g,u)=0$. Then $f=(9c-3g^2+6cu+cu^2)/(3+u)^2$ which implies the existence of the common factor $\psi=a(3+u)^3-g(9c-2g^2+6cu+cu^2)$ of the polynomials $\Gamma_i,\ i=1,2$. From $\psi=0$ it results $a=g(9c-2g^2+6cu+cu^2)/(3+u)^3$ and this gives $\Gamma_1=\Gamma_2=0$ and $\Gamma_3(c,g,u)=(9c-3g^2+6cu+g^2u+cu^2)^2[c(3+u)^2(3+2u)^2-g^2(27+27u+8u^2)]/(3+u)^8$. Since $\Phi=(9c-3g^2+6cu+g^2u+cu^2)^3/(3+u)^6\neq 0$ and in addition $(3+u)(3+2u)\neq 0$, the condition $\Gamma_3=0$ gives $c=g^2(27+27u+8u^2)/[(3+u)^2(3+2u)^2]$. Therefore considering the relations

$$f=-\frac{g^2u(9+4u)}{(3+u)^2(3+2u)^2},\quad a=\frac{3g^3}{(3+u)^2(3+2u)^2},\quad c=\frac{g^2(27+27u+8u^2)}{(3+u)^2(3+2u)^2} \tag{4.45}$$

with $g(2+u)\neq 0$ (otherwise we get degenerate systems) due to the transformation $(x,y,t)\mapsto(-g(3+2u+2ux)/[(3+u)(3+2u)],y,t(3+u)^2(3+2u)^2/[4g^2u^2])$ the last systems could be brought to the 1-parameter family of systems

$$\dot x=x(1+x)\big[u+2+(u+1)x\big],\quad \dot y=y\big[u+2+(3+2u)x+ux^2\big] \tag{4.46}$$

for which $H=X^3Y(X+Z)^2\big[(u+1)X+(u+2)Z\big]$.

Thus these systems possess 3 finite singularities: $(-1,0)$, $(0,0)$ and $(-(2+u)/(1+u),0)$ which are located on the invariant line $y=0$. On the other hand considering Lemma 2.1 for these systems we calculate: $\mu_0=\ldots=\mu_5=0,\quad \mu_6=(u+1)(2+u)^3x^6$. Since $(u+1)(2+u)\neq 0$ by Lemma 2.1 all other 6 finite singular points have gone to infinity and collapsed with the singular point $[0,1,0]$ located on the "end" of the invariant line $x=0$.

Thus a system (4.46) possesses four distinct invariant affine lines: three in the direction $x=0$ (one triple, one double and one simple) and one line in the direction $y=0$, and namely: $L_{1,2,3}=x,\ L_{4,5}=x+1,\ L_6=(u+1)x+(u+2),\ L_7=y$. Comparing the lines $x=-1$ and $x=-(u+2)/(u+1)$ with $x=0$ we detect the following possibilities which depend on the value of the parameter u:

- *The possibility* $u<-2$. Then the simple invariant line is located on the domain between the triple and the double ones and we obtain the configuration *Config. 8.44*;
- *The possibility* $-2<u<-1$. Then the triple invariant line is located on the domain between the simple and the double ones and we have *Config. 8.45*;

- *The possibility* $u > -1$. The double invariant line is located on the domain between the simple and the triple ones and we get the configuration *Config. 8.46.*

ii) Now we examine the condition $\Upsilon_2(c, f, g, u) = 0$. Then $f = -(6c - 2g^2 + 7cu + 2cu^2)/[(1+u)(3+2u)]$ and therefore we get

$$\Gamma_1(a, c, g, u) = [a(1+u)(3+2u)^3 - g(9c - 2g^2 + 12cu + 4cu^2)]^2/[(1+u)^2(3+2u)^3] = 0.$$

Since $(1+u)(3+2u) \neq 0$ the last condition gives $a = [g(9c-2g^2+12cu+4cu^2)]/[(1+u)(3+2u)^3]$ which implies $\Gamma_1 = \Gamma_2 = 0$ and

$$\Gamma_3 = 4(2+u)^2[\phi_1(c, g, u)]^3[c(3+u)^2(3+2u)^2 - g^2(27+27u+8u^2)]/[(1+u)(3+2u)^6],$$
$$\Phi = [\phi_1(c, g, u)]^2\phi_2(c, g, u)/[(1+u)(3+2u)^6] \neq 0$$

where ϕ_1 and ϕ_2 are polynomials of degree 3 and 4, respectively. Therefore the condition $\Gamma_3 = 0$ is equivalent to $c = g^2(27+27u+8u^2)/[(3+u)^2(3+2u)^2]$ and considering the above conditions we have $f = -\dfrac{g^2u(9+4u)}{(3+u)^2(3+2u)^2}$, $a = \dfrac{3g^3}{(3+u)^2(3+2u)^2}$ which all together are equivalent to the conditions (4.45).

iii) In the case $\Upsilon_3(c, f, g, u) = 0$ we suppose first $g \neq 0$. Then denoting $3f - cu + 3fu = c_1$ (i.e. $c = (3f + 3fu - c_1)/u$) the condition $\Upsilon_3 = 0$ gives $f = -c_1^2/(4g^2u)$. In this case the polynomial $\tilde{\psi} = c_1^3(1+u) + g^2u(c_1^2 - 4agu^2)$ is a common factor of Γ_1 and Γ_2 and it must vanish. However the calculations yield: $\Phi = \tilde{\psi}\tilde{\psi}_1/(g^6u^3) \neq 0$ (where $\tilde{\psi}_1$ is a polynomial) and we get a contradiction.

Assume now $g = 0$. In this case we get $\Upsilon_3 = (3f - cu + 3fu)^2 = 0$ which implies $c = 3f(1+u)/u$. Then the common factor of the polynomials Γ_1 and Γ_2 is $a^2u^3 + 4f^3(1+u)^2 = \Phi \neq 0$, i.e. we again arrive at a contradiction.

iv) Finally we suppose $\Upsilon_4(c, f, g, u) = 0$. Since Υ_4 is quadratic in c we must have $\text{Discriminant}[\Upsilon_4, c] = -16fg^2u^3(1+u)^2(2+u)^4 \geq 0$. It was proved earlier (see page 130) that in the case $u = -2$ the condition $\Gamma_1 = \Gamma_2 = 0$ leads to degenerate systems. So we assume $u + 2 \neq 0$ and we examine two subcases: $g \neq 0$ and $g = 0$.

If $g \neq 0$ then $fu \leq 0$ and we set a new parameter: $fu^3 = -v^2$ (i.e. $f = -v^2/u^3$) and we calculate $\Upsilon_4 = \Phi^{\pm} = 0$, where $\Phi^{\pm} = (1+u)(3+3u+u^2)v^2 \pm 2gu^2(1+u)v + u^4[c(2+u)^2 - g^2]$. It is clear that we could consider only the case $\Phi^- = 0$ (due to the change $v \to -v$) and this condition gives us $c = \left[g^2u^4 + 2gu^2(1+u)v - (1+u)(3+3u+u^2)v^2\right]/[u^4(2+u)^2]$. Substituting the expressions for the parameters f and c in the polynomials Γ_1 and Γ_2 we detect that the common factor of these polynomials is again $\Phi \neq 0$.

Suppose now $g = 0$. Then we have $\Upsilon_4 = f(1+u)(3+3u+u^2) - cu(2+u)^2$ and as $(1+u)(3+3u+u^2) \neq 0$ the condition $\Upsilon_4 = 0$ gives $f = cu(2+u)^2/[(1+u)(3+3u+u^2)]$ Herein we calculate $\Gamma_1 = (3+2u)^3\Phi/u^3$ and due to $\Phi \neq 0$ we obtain $\Gamma_1 \neq 0$. Therefore we proved that in the case either $\Upsilon_3 = 0$ or $\Upsilon_4 = 0$ systems (4.44) could not have invariant lines of total multiplicity eight.

2) The subcase $u+1=0$ Therefore we have $u=-1$ and systems (4.44) become

$$\dot{x} = a + cx + gx^2 \equiv \tilde{P}(x), \quad \dot{y} = y(f - x^2) \equiv y\tilde{Q}(x), \tag{4.47}$$

for which we calculate $H(X,Y,Z) = YZ(gX^2 + cXZ + aZ^2) \equiv Y\tilde{P}^*(X,Z)$. Moreover we have

$$\begin{aligned}
\mathcal{G}_1/H =& -X^4 + (c+2f)X^2Z^2 + 2(a+fg)XZ^3 + (c-f)fZ^4 \equiv F_1(X,Z),\\
\mathcal{G}_2/H =& -\big[2X^3 - (c+2f)XZ^2 - (a+fg)Z^3\big]\tilde{P}^*(X,Z) \equiv F_2(X,Z)\tilde{P}^*(X,Z),
\end{aligned}$$

and $\mathcal{G}_3/H = 24Z^2[\tilde{Q}^*(X,Z)]\big[\tilde{P}^*(X,Z))\big]^2$. So we need to determine a common factor of degree 3 of the polynomials $\mathcal{G}_i/H$, $i=1,2,3$, which in fact must contains only the factors of the polynomial $\tilde{P}^*(X,Z)$ (see Remark 3.4).

On the other hand we observe that for non-degenerate systems (4.47) the polynomials $f - x^2$ and $\tilde{P}(x)$ have no common factors, i.e. the following condition must holds: $\Phi(a,c,f,g) \equiv R_x^{(0)}(\tilde{P}(x),\tilde{Q}(x)) = (a+fg)^2 - c^2 f \neq 0$. Thus considering the structure of the polynomial $\mathcal{G}_3/H$ we deduce that the polynomial $\tilde{P}^*(X,Z)$ must be a factor of the polynomial F_1. So the following conditions are necessary: $R_X^{(0)}\Big(F_1, \tilde{P}^*(X,Z)\Big) = R_X^{(1)}\Big(F_1, \tilde{P}^*(X,Z)\Big) = 0$. We calculate

$$R_X^{(1)}\Big(F_1, \tilde{P}^*(X,Z)\Big) = (c-g^2)(2ag - c^2 + 2fg^2)Z^3 = 0 \tag{4.48}$$

and we observe that $c - g^2 \neq 0$, otherwise supposing $c = g^2$ and considering (4.48) we obtain $R_X^{(0)}\Big(\mathcal{G}_1/H, \tilde{P}^*(X,Z)\Big) = \Phi^2 Z^8 \neq 0$. Thus for non-degenerate systems the condition (4.48) gives $2ag - c^2 + 2fg^2 = 0$, where $g \neq 0$, otherwise we get $g = c = 0$ which contradicts $c - g^2 \neq 0$. So we obtain $a = c^2 - 2fg^2/(2g)$ and calculations yield $R_X^{(0)}\Big(F_1, \tilde{P}^*(X,Z)\Big) = c^2(c-2g^2)^2(c^2-4fg^2)^2Z^8/(16g^4) = 0$, $\Phi = c^2(c^2-4fg^2)/(4g^2) \neq 0$. Therefore $c = 2g^2$ and considering (4.43) and the relations $u=-1$ and $a = (c^2-2fg^2)/(2g)$ we calculate $R_X^{(0)}\Big(F_1(X,Z)/\tilde{P}^*, \tilde{P}^*(X,Z)\Big) = 4g^2(4f-5g^2)Z^4$, $\psi = -82g(4f-5g^2) = 0$. Hence due to $g \neq 0$ we get the unique condition $f = 5g^2/4$ and this leads to the family of systems $\dot{x} = g(g+2x)(3g+2x)/4$, $\dot{y} = y(5g^2 - 4x^2)/4$, which via the changing $(x,y,t) \mapsto (g(x - 1/2), y, t/g^2)$ could be brought to the system

$$\dot{x} = x(1+x), \quad \dot{y} = y(1+x-x^2). \tag{4.49}$$

For this system we have $H(X,Y,Z) = X^3YZ(X+Z)^2$, i.e. it belongs to $\mathbb{CSL}_8$. We observe that system (4.49) has 2 finite singularities: $(-1,0)$, $(0,0)$ which are located on the invariant line $y=0$. On the other hand considering Lemma 2.1 for these systems we calculate: $\mu_0 = \ldots = \mu_6 = 0$, $\mu_7 = -x^6y$. By Lemma 2.1 all other 7 finite singular points have gone to infinity. Moreover, according to this lemma, six of them collapsed with the singular point

$[0,1,0]$ located on the "end" of the invariant line $x=0$ and the remaining one collapses with the singular point $[1,0,0]$ located on the "end" of the invariant line $y=0$. Thus besides the double infinite line a system (4.49) possesses three distinct invariant affine lines: two in the direction $x=0$ (one triple and one simple) and a line in the direction $y=0$, and namely: $L_{1,2,3}=x,\ L_4=x+1,\ L_5=y$. Therefore we get the configuration *Config. 8.47*.

In such a way taking into account our article [17] we have proved the next result.

Proposition 4.7. *Systems* (4.42) *possess invariant lines of total multiplicity eight if and only if the following set of conditions holds:*

$$e=b=0, a=\frac{3g^3}{(3+u)^2(3+2u)^2}, c=\frac{g^2(27+27u+8u^2)}{(3+u)^2(3+2u)^2}, f=\frac{-g^2u(9+4u)}{(3+u)^2(3+2u)^2}, g(u+2)\neq 0. \quad (4.50)$$

4.2. Invariant criteria for the realization of the configurations with two distinct infinite singularities

By Lemma 2.2 the conditions $\mathcal{D}_1=\mathcal{D}_3=\mathcal{D}_4=0,\ \mathcal{D}_2\neq 0$ are necessary and sufficient for a cubic system to have two real distinct infinite singularities, and namely they are determined by one triple and one simple factors of $C_3(x,y)$. After a linear transformation a cubic system could be brought to the form (4.1). According to Proposition 4.1 the above mentioned cubic systems could have one of the four cubic homogeneities given by this lemma. Since for the homogeneity $(2x^3,3x^2y)$ we get no configurations (see [17]) we restrict our attention to the remaining three cubic homogeneities: $(x^3,0)$, $(x^3,3x^2y)$ and $\big((u+1)x^3,ux^2y\big)$.

I. Conditions for *Config. 8.23* – *Config. 8.38*. According to Proposition 4.1, the condition $\mathcal{V}_1=\mathcal{V}_3=0$ gives systems (4.2) (via a linear transformation and time rescaling). By Remark 4.1 for these systems the condition $\mathcal{K}_5=0$ is equivalent to $k=h=n=0$. Moreover, for the existence of invariant lines in the direction $x=0$ the additional condition $d=0$ has to be satisfied. So considering the condition $\mathcal{K}_5=0$ for systems (4.2) we calculate $N_1=12d$ and evidently $N_1=0$ is equivalent to $d=0$ and we arrive at systems (4.3). For these systems we calculate $N_2=-m^2x^4,\ N_3=-12x^5l$. We remark that in the previous subsections the examination of systems (4.3) was divided in the cases determined by the parameters m and l. In addition it was proved earlier that in the case $m=0$ and $l\neq 0$ (i.e. $N_2=0$ and $N_3\neq 0$) systems (4.3) could not have invariant lines of total multiplicity 8. So in what follows we split our examination here in three cases, defined by the invariant polynomials N_2 and N_3:

$$(i)\ N_2N_3\neq 0;\quad (ii)\ N_2\neq 0, N_3=0;\quad (iii)\ N_2=N_3=0.$$

1) The case $\boldsymbol{N_2N_3\neq 0}$ Then $l\,m\neq 0$ and as it was shown earlier systems (4.3) could be brought via an affine transformation to systems (4.8). According to Proposition 4.2 the last systems belong to $\mathbb{CSL}_8$ if and only if the conditions (4.2) are satisfied. We prove that these conditions are equivalent to $N_4 = N_5 = N_6 = N_7 = 0$, i.e.

$$a = 0,\ f = c = -\frac{4}{9},\ b = \frac{4}{27} \quad \Leftrightarrow \quad N_4 = N_5 = N_6 = N_7 = 0.$$

Indeed, for systems (4.8) we calculate $N_4 = 5184(c-f)x^4$ and $N_5 = 2592(4+6c+3f)x^4$ and clearly the condition $N_4 = N_5 = 0$ is equivalent to $f = c = -4/9$. Then considering the last conditions we calculate $N_6 = 8640ax^4$ and hence $N_6 = 0$ gives $a = 0$. It remains to determine the invariant polynomial which governs the condition for the parameter b. Considering the obtained conditions for systems (4.8) we calculate $N_7 = 288(27b-4)x^6 = 0$ which is equivalent to $b = \dfrac{4}{27}$. So if for systems (4.3) the conditions $N_2N_3 \neq 0,\ N_4 = N_5 = N_6 = N_7 = 0$ are satisfied then we arrive at the system (4.10) possessing the configuration *Config. 8.23*.

2) The case $N_2 \neq 0,\ N_3 = 0$. These conditions imply $m \neq 0$ and $l = 0$, and as it was proved that in this case the condition $ef - 2bm = 0$ is necessary to be fulfilled for systems (17) in order to have invariant lines of total multiplicity 8. On the other hand for these systems we calculate $N_8 = 1296(ef-2bm)x^6$ and the last condition is equivalent to $N_8 = 0$. Due to a rescaling we may assume $m = 1$ and then we get $b = ef/2$ and this leads to systems (4.12). By Proposition 4.3 these systems belong to $\mathbb{CSL}_8$ if and only if either the conditions (4.13) or (4.14) are satisfied.

In what follows we consider each one of these sets of conditions and construct the corresponding equivalent invariant conditions as well as the additional invariant conditions for the realization of the respective configurations.

(a) Conditions (4.13). We clam that for a system (4.12) the following conditions are equivalent:

$$f = c,\ a = -\frac{2(4+9c)}{27},\ (4+3c)(4+9c) \neq 0 \quad \Leftrightarrow \quad N_4 = N_6 = 0,\ N_9 \neq 0.$$

Indeed, for systems (4.12) we calculate $N_4 = 5184(c-f)x^4$ and therefore $N_4 = 0$ gives $f = c$. Then we have $N_6 = 320(27a + 18c + 8)x^4 = 0$ and $N_9 = 2304(4+3c)(4+9c)x^4 \neq 0$ which imply the condition $a = -2(4+9c)/27$.

Thus if the conditions $N_4 = N_6 = 0$ are satisfied then systems (4.12) via a translation and a suitable notation can be brought to systems (4.15), for which the condition $N_9 = 6912r(9r-8)x^4 \neq 0$ holds. Now for these systems we need to determine the invariant polynomials which govern the conditions under parameter r in order to get different configurations of invariant straight lines. We calculate $N_{10} = 144(1-r)x^2$ and $N_{11} = 3456rx^4$. Therefore, considering the obtained earlier for systems (4.15) configurations (see page 118) we conclude that if for a system (4.3) the conditions $N_3 = N_4 = N_6 = N_8 = 0,\ N_2N_9 \neq 0$ are satisfied then we get the configuration given by *Config. 8.24* if $N_{11} < 0$; by *Config. 8.25* if $N_{10} > 0$ and $N_{11} > 0$; by *Config. 8.26* if $N_{10} = 0$ and by *Config. 8.27* in the case $N_{10} < 0$.

(b) Conditions (4.14). We clam that for a system (4.12) the next conditions are equivalent:

$$f = -\frac{2(2+3c)}{3},\ a = \frac{2(4+9c)}{27},\ (4+3c)(4+9c) \neq 0 \quad \Leftrightarrow \quad N_5 = N_{12} = 0,\ N_{13} \neq 0.$$

Indeed, for (4.12) we calculate $N_5 = 2592(4+6c+3f)x^4$ and hence $N_5 = 0$ implies $f = -2(2+3c)/3$. Then we have $N_{12} = 3240(27a-18c-8)x^4$ and, clearly, $N_{12} = 0$ gives $a = 2(4+9c)/27$. For $N_5 = N_{12} = 0$ we calculate $N_{13} = 1008(4+3c)(4+9c)x^5y$ and therefore $N_{13} \neq 0 \Leftrightarrow (4+3c)(4+9c) \neq 0$.

So, considering the above relations among the parameters a, c and f of systems (4.12) it was shown earlier that these systems can be brought (via a translation and additional notation) to systems (4.16).

It remains to determine the invariant polynomial which gives the expression of the discriminant $\Delta = 4(1-r)$. For these systems we calculate $N_{14} = 288(r-1)x^2$ and $N_{15} = 2^9 3^7 r x^4$.

Therefore if for a system (4.3) the conditions $N_3 = N_5 = N_8 = N_{12} = 0,\ N_2N_{13} \neq 0$ are satisfied then we get *Config. 8.28* if $N_{15} < 0$; *Config. 8.29* if $N_{14} < 0,\ N_{15} > 0$ and *Config. 8.30* if $N_{14} > 0$.

3) The case $N_2 = N_3 = 0$. Then $l = m = 0$ and we get systems for which we calculate $N_{16} = -12ex^4$. In what follows we split our examination here in two subcases, defined by the polynomial N_{16}.

a) *The subcase* $N_{16} \neq 0$. Then $e \neq 0$ and systems (4.3) could be brought via a rescaling (i.e. assuming $e = 1$) to systems (4.19). According to Proposition 4.4 the last systems belon to $\mathbb{CSL}_8$ if and only if the conditions (4.20) are satisfied. We prove that these conditions are equivalent to $N_{17} = N_{18} = 0$, i.e. $f = -2c,\quad a = 0 \ \Leftrightarrow\ N_{17} = N_{18} = 0$.
Indeed, for the corresponding systems we calculate $N_{17} = 12(2c+f)x^2 = 0,\ N_{18} = 216ax^3 = 0$ and evidently, the above equalities are equivalent to $f = -2c,\ a = 0$.

It remains to determine the invariant condition which governs the value of c. For the last systems we determine $N_{10} = 72cx^2$. Next we split our examinations according to the parameter c.

i) *The possibility* $N_{10} \neq 0$. Then $c \neq 0$ and assuming $b = 0$ after a translation we arrive at the system (4.22). So, if for systems (4.3) the conditions $N_2 = N_3 = N_{17} = N_{18} = 0,\ N_{10}N_{16} \neq 0$ are satisfied then we get the configuration *Config. 8.31* if $N_{10} < 0$ and *Config. 8.32* if $N_{10} > 0$.

ii) *The possibility* $N_{10} = 0$. Then $f = c = 0$ and after a rescaling we assume $b = 1$ and we get the systems (4.21). So, if for systems (4.3) the conditions $N_2 = N_3 = N_{10} = N_{17} = N_{18} = 0,\ N_{16} \neq 0$ are satisfied then we get the configuration *Config. 8.33*.

b) *The subcase* $N_{16} = 0$. Then $e = 0$ and systems (4.3) became of the form (4.23).

According to Proposition 4.5 the last systems belong to $\mathbb{CSL}_8$ if and only if the conditions (4.27) hold. We prove that these conditions are equivalent to $N_{19} = 0,\ N_{18} \neq 0$, i.e.

$$27a^2 + (c - f)(2c + f)^2 = 0,\ a \neq 0\ \Leftrightarrow N_{19} = 0,\ N_{18} \neq 0.$$

Indeed, for systems (4.23) we have $N_{19} = 24[27a^2+(c-f)(2c+f)^2]x^3y$ and, evidently, $N_{19} = 0$ implies $27a^2 + (c-f)(2c+f)^2 = 0$. On the other hand we have $N_{18} = 216ax^3$ and thus, the condition $N_{18} \neq 0$ is equivalent to $a \neq 0$. Therefore if the conditions $N_{19} = 0,\ N_{18} \neq 0$ are satisfied then systems (4.23) via a transformation and a suitable notation (see page 123) can be brought to systems (4.25). For these systems we calculate $N_{20} = 48(1 - 4r)x^4,\ \ N_{21} = 48rx^4$.

Therefore if for a system (4.3) the conditions $N_2 = N_3 = N_{16} = N_{19} = 0$ and $N_{18} \neq 0$ hold then we obtain the configuration *Config. 8.34* if $N_{21} < 0$; *Config. 8.35* if $N_{20} > 0, N_{21} > 0$; *Config. 8.36* if $N_{20} = 0$ and *Config. 8.37* in the case $N_{20} < 0$. Moreover if $N_{21} = 0$, i.e. $r = 0$ we obtain *Config. 8.38*.

II. Conditions for *Config.8.39-Config.8.47.* So by Proposition 4.1 the conditions $\mathcal{V}_1 = 0,\ \mathcal{V}_3 \neq 0$ and respectively $\mathcal{V}_5 = \mathcal{U}_2 = 0,\ \mathcal{V}_1\mathcal{V}_3 \neq 0$ applying to systems (4.1) lead us to systems (4.28) and respectively (4.41) (via a linear transformation and time rescaling). Additionally for a system (4.28) (respectively (4.41)) we applied Remark 3.1 and we proved that the condition $\mathcal{L}_1 = \mathcal{L}_2 = 0$ (respectively $\mathcal{K}_4 = \mathcal{K}_5 = 0$) is equivalent to $n = k = h = d = 0$ which leads to systems (4.29) (respectively (4.42)).

In what follows considering systems (4.29) and (4.42) we find out the invariant conditions which are equivalent to the conditions mentioned in Propositions 4.6 and 4.7.

1) **Conditions for systems** (4.29). For these systems we calculate $N_{22} = mx^5$ and it is evident that the condition $N_{22} = 0$ is equivalent to $m = 0$ and in this case we calculate

$$N_{23} = -3ex^6 + 3(c - f)x^5y,\quad N_{24} = 216bx^{13},\quad \mathcal{K}_6 = 162000ax^{11}.$$

Thus $N_{23} = 0$ implies $e = c - f = 0$ and therefore, according to Proposition 4.6, next we split our examination in two cases: $b = 0,\ a \neq 0$ i.e. $N_{24} = 0,\ \mathcal{K}_6 \neq 0$ and $b \neq 0,\ a = 0$, i.e. $N_{24} \neq 0,\ \mathcal{K}_6 = 0$.

a) If $N_{24} = 0$ and $\mathcal{K}_6 \neq 0$ then we arrive at systems (4.34) for which $\mu_6 = (27a^2 + 4c^3)x^6$ should be non-zero in order to have non-degenerate systems. Moreover due to a transformation the last systems became of the form (4.36) with $\mu_6 = r^2(4r - 1)x^6 \neq 0$.

So if for systems (4.29) the conditions $N_{22} = N_{23} = N_{24} = 0,\ \mathcal{K}_6 \neq 0$ hold then we get either *Config. 8.39* if $\mu_6 < 0$ or *Config. 8.40* if $\mu_6 > 0$.

b) Assume now $N_{24} \neq 0$ and $\mathcal{K}_6 = 0$. Then applying a rescaling we arrive at systems (4.39) for which $\mu_6 = 4c^3x^6 \neq 0,\ c = \{-1, 0, 1\}$. Therefore if for a system (4.42) the conditions

$N_{22} = N_{23} = \mathcal{K}_6 = 0$, $N_{24} \neq 0$ are satisfied then we have *Config. 8.41* if $\mu_6 < 0$; *Config. 8.42* if $\mu_6 = 0$ and *Config. 8.43* if $\mu_6 > 0$.

2) **Conditions for systems** (4.42). For these systems we calculate $N_{24} = 2bu^4x^{13}/3$, $N_{25} = 5eu^2x^{10}/3$. It is evident that the condition $N_{24} = N_{25} = 0$ implies $b = e = 0$ (since $u \neq 0$, see Lemma 4.1). In this case we have $N_{26} = 20u^4(9a - cg - 2fg + 3au)x^{10}y/9$ and

$$N_{27} = 40u^4[3(c-f)(12c+15f-4g^2) - 3u(c-f)^2 - 16g^2u(c+2f) + 4(c-f)(c+2f)u^2]/9.$$

Taking into consideration the expression of $\mathcal{K}_6$ (see (4.43)), it is easy to verify that $N_{26} = N_{27} = \mathcal{K}_6 = 0$ lead us to the conditions (4.50). Since $u(3+u)(3+2u) \neq 0$ (due to $\mathcal{V}_1\mathcal{V}_3 \neq 0$) and $g \neq 0$ (as systems are non-degenerate) applying the corresponding transformation (mentioned on page 131) to systems (4.42) with the conditions (4.50) we arrive at systems (4.46) for which we have $u+2 \neq 0$ (otherwise these systems become degenerate). For systems (4.46) we calculate $\mu_i = 0$, $\mu_6 = (u+1)(u+2)^3x^6$, $i = 0, 1, \ldots, 5$ and $N_{28} = -2(3+2u)x^4$. If $\mu_6 \neq 0$ we obtain $\text{sign}\,(\mu_6) = \text{sign}\,\big((u+1)(u+2)\big)$. Therefore if $\mu_6 < 0$ then $-2 < u < -1$ and we get *Config. 8.44*, whereas in the case $\mu_6 > 0$ we have either *Config. 8.45* for $N_{28} < 0$ or *Config. 8.46* for $N_{28} > 0$. Additionally if $\mu = 0$, i.e. $u = -1$ then we we arrive at systems (4.49) and we get *Config. 8.47*.

4.3. Perturbations of canonical forms

To finish the proof of the Main Theorem D it remains to construct for the normal forms given in this theorem the corresponding perturbations, which prove that the respective invariant straight lines have the indicated multiplicities. In this section we construct such perturbations and for each configuration *Configs. 8.j*, $j = 23, 40, \ldots, 47$ we give:

- the corresponding normal form and its invariant straight lines;
- the respective perturbed normal form and its invariant straight lines;
- the configuration *Config.* $8.j_\varepsilon$, $j = 23, 24, \ldots, 47$ which corresponds to the perturbed systems.

Config. 8.23 $\begin{cases} \dot{x} = (x-1)x(1+x), \\ \dot{y} = x - y + x^2 + 3xy; \end{cases}$

Invariant lines: $L_{1,2} = x$, $L_{3,4,5} = x - 1$, $L_6 = x + 1$, $L_7 : Z = 0$;

Config. 8.23$_\varepsilon$: $\begin{cases} \dot{x} = x(1+x)(x + 3x\varepsilon - 1), \\ \dot{y} = (1+3\varepsilon y)(x + x^2 - y + 3xy - 3\varepsilon y + 3\varepsilon xy - 6\varepsilon y^2 - 9\varepsilon^2 y^2); \end{cases}$

Invariant lines: $\begin{cases} L_1 = x, \quad L_2 = x - 3\varepsilon y, \quad L_3 = x + 3\varepsilon x - 1, \ L_4 = x - 3\varepsilon y - 1, \\ L_5 = x - 3\varepsilon - 6\varepsilon y - 9\varepsilon^2 y - 1, \ L_6 = 1 + x, \ L_7 = 1 + 3\varepsilon y. \end{cases}$

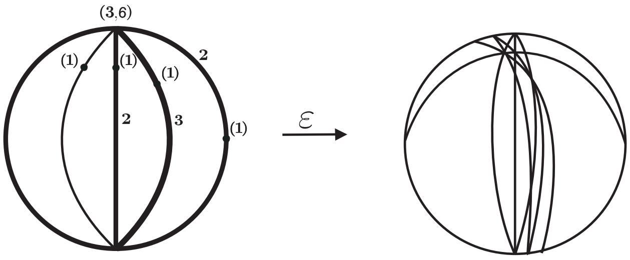

Fig. 4.2. Perturbation of normal form corresponding to the configuration *Config. 8.23*

Config. 8.24-8.26 $\begin{cases} \dot{x} = x(1-u+x)(1+u+x), \\ \dot{y} = (1-u^2+2x)y, \quad |u| \neq 1, \end{cases}$ $\begin{cases} |u| > 1 & \Rightarrow \textit{Config. 8.24}; \\ |u| < 1 & \Rightarrow \textit{Config. 8.25}; \\ u = 0 & \Rightarrow \textit{Config. 8.26}; \end{cases}$

Invariant lines: $L_{1,2,3} = x,\ L_4 = x+1+u,\ L_5 = x+1-u,\ L_6 = y,\ L_7:\ Z = 0;$

Config. 8.24$_\varepsilon$-8.26$_\varepsilon$: $\begin{cases} \dot{x} = x(1-u+\varepsilon^2+x)(1+u-\varepsilon^2+x), \\ \dot{y} = y(1+\varepsilon y)\big[1-(u-\varepsilon^2)^2+2x+\big(\varepsilon^2(u-\varepsilon^2)^2-\varepsilon^2\big)y\big]; \end{cases}$

Invariant lines: $\begin{cases} L_1 = x, \quad L_2 = x-\varepsilon(u+1)y, \quad L_3 = x-\varepsilon(u-1)y-y\varepsilon^3, \\ L_4 = x+1+u-\varepsilon^2,\ L_5 = x+1-u+\varepsilon^2,\ L_6 = y,\ L_7 = 1+\varepsilon y. \end{cases}$

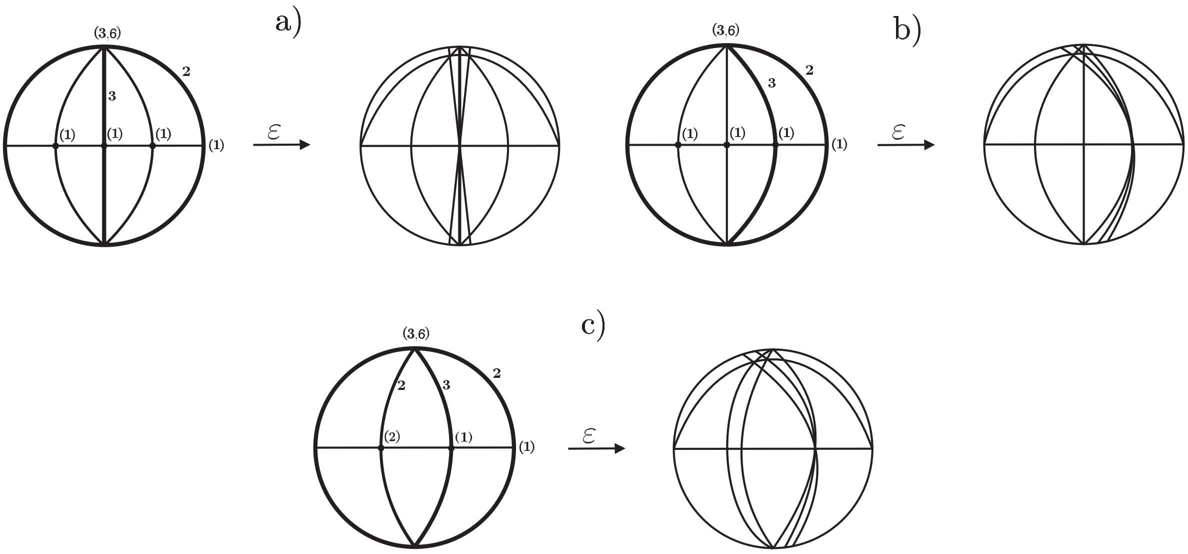

Fig. 4.3. Perturbations of normal forms corresponding to the configurations: a) *Config. 8.24*; b) *Config. 8.25*; c) *Config. 8.26*

Config. 8.27: $\begin{cases} \dot{x} = x\big[(x+1)^2+u^2\big], \\ \dot{y} = (1+u^2+2x)y, \quad u \neq 0; \end{cases}$

Invariant lines: $L_{1,2,3} = x,\ L_4 = x+1+iu,\ L_5 = x+1-iu,\ L_6 = y,\ L_7:\ Z = 0;$

Config. 8.27$_\varepsilon$: $\begin{cases} \dot{x} = x\big[(x+1)^2 + u^2\big], \\ \dot{y} = y(1 - y\varepsilon)(1 + u^2 + 2x + y\varepsilon + u^2 y\varepsilon); \end{cases}$

Invariant lines: $L_1 = x,\ L_{2,3} = x + \varepsilon y \pm iu\varepsilon y,\ L_{4,5} = x + 1 \pm iu,\ L_6 = y,\ L_7 = -1 + y\varepsilon.$

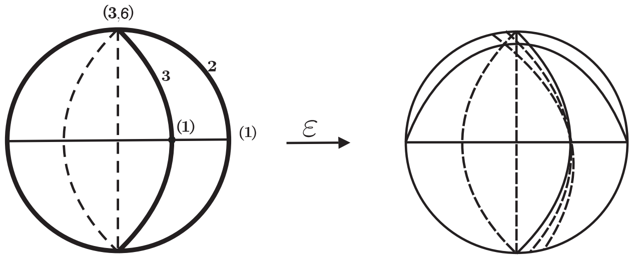

Fig. 4.4. Perturbation of normal form corresponding to the configuration *Config. 8.27*

Config. 8.28, 8.29 $\begin{cases} \dot{x} = (1 - x + u)x(1 - x - u), \\ \dot{y} = 2(u^2 + x - 1)y,\ |u| \neq 1, \end{cases}$ $\begin{cases} |u| > 1 \ \Rightarrow\ \textit{Config. 8.28}; \\ |u| < 1 \ \Rightarrow\ \textit{Config. 8.29}; \end{cases}$

Invariant lines: $L_1 = x,\ L_{2,3} = 1 - x + u,\ L_{4,5} = 1 - x - u,\ L_6 = y,\ L_7:\ Z = 0;$

Config. 8.28$_\varepsilon$, 8.29$_\varepsilon$: $\begin{cases} \dot{x} = (1 - x + u)x(1 - x - u), \\ \dot{y} = y(1 + u - \varepsilon y)(2u^2 + 2x + \varepsilon y - u\varepsilon y - 2)/(1 + u); \end{cases}$

Invariant lines: $\begin{cases} L_1 = x,\quad L_2 = 1 - x + u,\quad L_3 = 1 - x + u - \varepsilon y,\ L_4 = 1 - x - u, \\ L_5 = x - 1 + u^2 + ux + \varepsilon y - u\varepsilon y,\ L_6 = y,\ L_7 = 1 + u - \varepsilon y. \end{cases}$

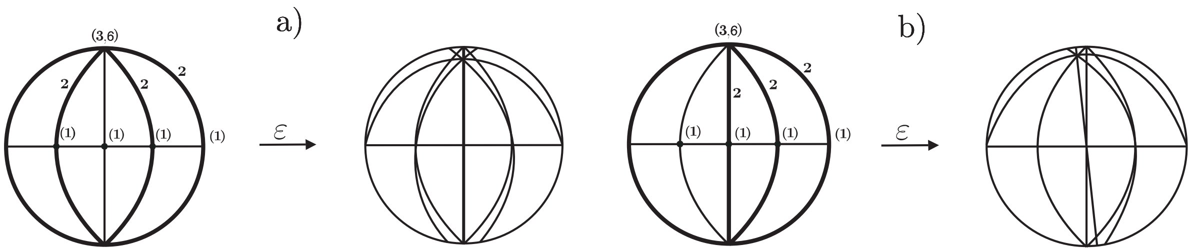

Fig. 4.5. Perturbations of normal forms corresponding to the configurations: a) *Config. 8.28*; b) *Config. 8.29*

Config. 8.30: $\begin{cases} \dot{x} = x(1 + u^2 - 2x + x^2), \\ \dot{y} = 2y(x - 1 - u^2), \quad u \neq 0; \end{cases}$

Invariant lines: $L_1 = x,\ L_{2,3} = x - 1 - iu,\ L_{4,5} = x - 1 + iu,\ L_6 = y,\ L_7:\ Z = 0;$

Config. 8.30$_\varepsilon$: $\begin{cases} \dot{x} = x(1 + u^2 - 2x + x^2), \\ \dot{y} = y(1 - \varepsilon y)(2x - 2 - 2u^2 + \varepsilon y + u^2\varepsilon y); \end{cases}$

Invariant lines: $\begin{cases} L_1 = x,\ L_2{=}x-1-iu,\ L_3{=}x-1-iu+y\varepsilon+iu\varepsilon y, \\ L_4{=}x{-}1{+}iu,\ L_5{=}x{-}1{+}iu{+}\varepsilon y{-}iu\varepsilon y,\ L_6 = y,\ L_7{=}\varepsilon y{-}1. \end{cases}$

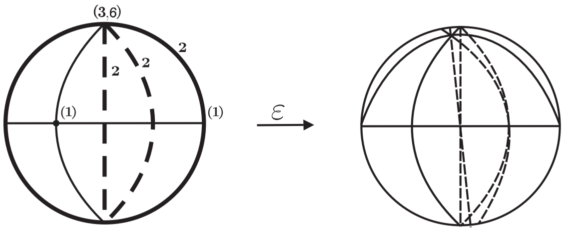

Fig. 4.6. Perturbation of normal form corresponding to the configuration *Config. 8.30*

Config. 8.31, 8.32 $\begin{cases} \dot x = x(x^2+r), \\ \dot y = x-2ry, \end{cases}$ $\begin{cases} r=-1 & \Rightarrow \textit{Config. 8.31}; \\ r=1 & \Rightarrow \textit{Config. 8.32}; \end{cases}$

Invariant lines: $L_1 = x,\ L_{2,3}{=}x{-}\sqrt{-r},\ L_{4,5}{=}x{+}\sqrt{-r},\ L_6{=}y,\ L_{6,7}: Z{=}0;$

Config. 8.31$_\varepsilon$, 8.32$_\varepsilon$: $\begin{cases} \dot x = (2r-\varepsilon^4+\varepsilon^6)(4r+4x^2-4r\varepsilon^2-3\varepsilon^4+6\varepsilon^6-3\varepsilon^8)\times \\ (x-x\varepsilon+6ry\varepsilon+2ry\varepsilon^2-3y\varepsilon^5-y\varepsilon^6+3y\varepsilon^7+y\varepsilon^8)/(8r), \\ \dot y = (x-2ry+\varepsilon^4y-y\varepsilon^6)(4r-4r\varepsilon^2+16r^2\varepsilon^2y^2-3\varepsilon^4+ \\ +6\varepsilon^6{-}16r\varepsilon^6y^2{-}3\varepsilon^8{+}16r\varepsilon^8y^2{+}4\varepsilon^{10}y^2{-}8\varepsilon^{12}y^2{+}4\varepsilon^{14}y^2)/(4r); \end{cases}$

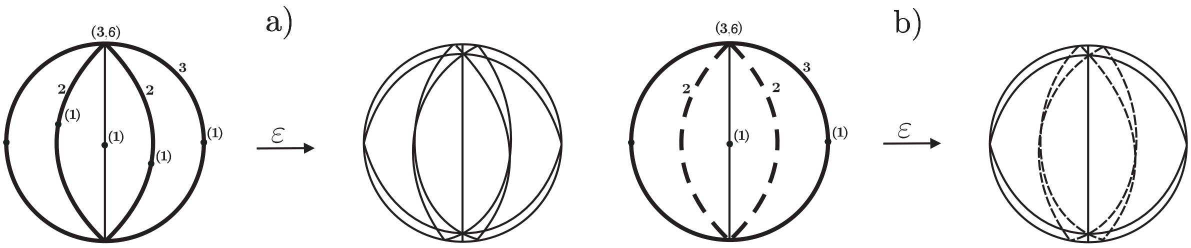

Fig. 4.7. Perturbations of normal forms corresponding to the configurations: a) *Config. 8.31*; b) *Config. 8.32*

Config. 8.33: $\left\{ \dot x = x^3, \quad \dot y = 1+x; \right.$

Invariant lines: $L_{1,2,3,4,5} = x,\ L_{6,7}:\ Z=0;$

Config. 8.33$_\varepsilon$: $\begin{cases} \dot x = x(9x-6\varepsilon+4\varepsilon^2)(9x+6\varepsilon-10\varepsilon^2+4\varepsilon^3)/81, \\ \dot y = (3{-}2\varepsilon{+}y\varepsilon^2)(3{-}2\varepsilon{-}y\varepsilon^2)(9{+}9x{-}15\varepsilon{+}6\varepsilon^2{-}\varepsilon^2y{+}\varepsilon^3y)/81; \end{cases}$

Invariant lines: $\begin{cases} L_1 = x,\ L_2 = x-6\varepsilon+4\varepsilon^2,\ L_3 = x+6\varepsilon-10\varepsilon^2+4\varepsilon^3, \\ L_4 = x-3\varepsilon+2\varepsilon^2+\varepsilon^3y,\ L_5 = x{+}3\varepsilon{-}5\varepsilon^2{+}2\varepsilon^3{-}\varepsilon^3y{+}\varepsilon^4y, \\ L_6 = 3-2\varepsilon+\varepsilon^2y,\ L_7 = -3+2\varepsilon+\varepsilon^2y. \end{cases}$

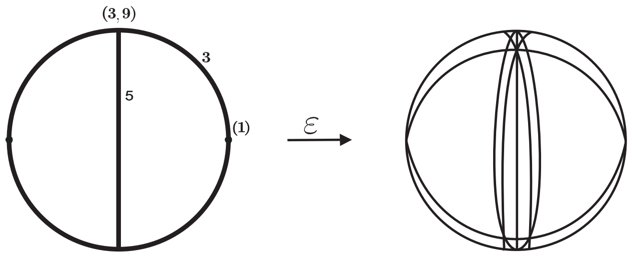

Fig. 4.8. Perturbation of normal form corresponding to the configuration *Config. 8.33*

Config. 8.34-8.37 $\begin{cases} \dot{x} = x(r + x + x^2), \\ \dot{y} = ry, \quad r \neq 0, \end{cases}$ $\begin{cases} r < 0 & \Rightarrow \textit{Config. 8.34}; \\ 0 < r < 1/4 & \Rightarrow \textit{Config. 8.35}; \\ r = 1/4 & \Rightarrow \textit{Config. 8.36}; \\ r > 1/4 & \Rightarrow \textit{Config. 8.37}; \end{cases}$

Invariant lines: $L_{1,2} = x,\ L_{3,4} = r + x + x^2,\ L_5 = y,\ L_{6,7}:\ Z = 0;$

Config. 8.34$_\varepsilon$*-8.37*$_\varepsilon$: $\begin{cases} \dot{x} = x(r - \varepsilon^2 + x + x^2), \\ \dot{y} = y(r - \varepsilon^2 - \varepsilon y + \varepsilon^2 y^2); \end{cases}$

Invariant lines: $\begin{cases} L_1 = x, \quad L_2 = x - \varepsilon y, \quad L_{3,4} = r + x + x^2 - \varepsilon^2, \\ L_{5,6} = r - \varepsilon y - \varepsilon^2 + \varepsilon^2 y^2,\ L_7 = y. \end{cases}$

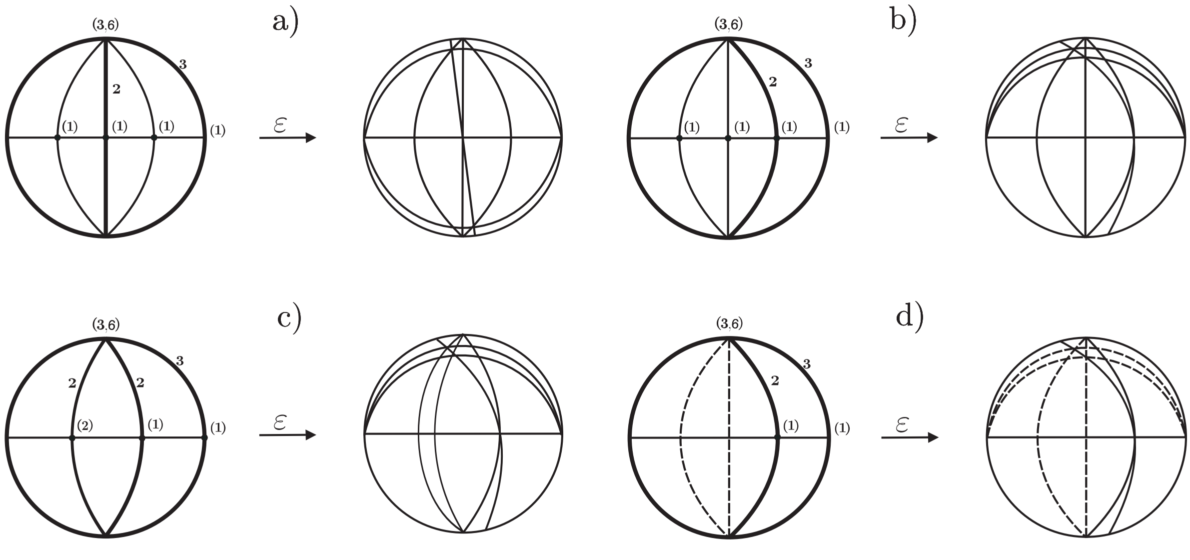

Fig. 4.9. Perturbations of normal forms corresponding to the configurations: a) *Config. 8.34*; b) *Config. 8.35*; c) *Config. 8.36*; d) *Config. 8.37*

Config. 8.38: $\left\{ \dot{x} = x^2(x + 1), \quad \dot{y} = 1; \right.$

Invariant lines: $L_{1,2,3} = x,\ L_4 = x + 1,\ L_{5,6,7}:\ Z = 0;$

Config. 8.38$_\varepsilon$: $\begin{cases} \dot{x} = x(x-\varepsilon)(1+x+\varepsilon-2\varepsilon y), \\ \dot{y} = (\varepsilon y-1)(2\varepsilon y-1)(1-2\varepsilon y+2\varepsilon^2 y); \end{cases}$

Invariant lines: $\begin{cases} L_1 = x,\ L_2 = x-\varepsilon,\ L_3 = x+\varepsilon-2y\varepsilon^2, L_4 = 1+x-\varepsilon-2y\varepsilon+2y\varepsilon^2, \\ L_5 = y\varepsilon-1,\ L_6 = 2y\varepsilon-1,\ L_7 = 1-2y\varepsilon+2y\varepsilon^2. \end{cases}$

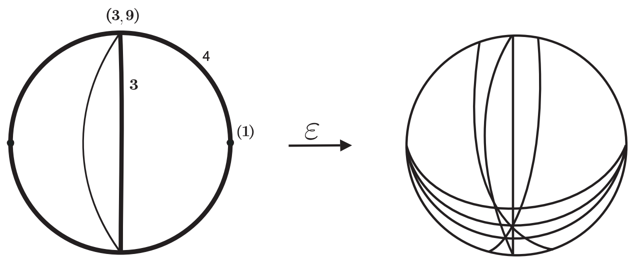

Fig. 4.10. Perturbation of normal form corresponding to the configuration *Config. 8.38*

Config. 8.39: $\begin{cases} \dot{x} = x(1+x)(v+x), \\ \dot{y} = (v+2x+2vx+3x^2)y; \end{cases}$

Invariant lines: $L_{1,2} = x,\ L_{3,4} = x+1,\ L_{5,6} = x+v,\ L_7 = y;$

We note that these systems are obtained from (4.37) due to the transformation $(x, y, t) \mapsto \big((1-u)x/2, y, 4t/(u-1)\big)$ and notation $v = (1+u)/(1-u)$, where $v \neq \{0, 1\}$ (since $u(1-u^2) \neq 0$).

Config. 8.39$_\varepsilon$: $\begin{cases} \dot{x} = x(1+x)(v+x), \\ \dot{y} = y[v+2x+2vx+3x^2+\varepsilon(y+vy+3xy+y^2\varepsilon)]; \end{cases}$

Invariant lines: $\begin{cases} L_1 = x,\ L_2 = x+y\varepsilon,\ L_3 = x+1,\ L_4 = x+1+y\varepsilon,\ L_5 = x+v, \\ L_6 = x+v+y\varepsilon,\ L_7 = y. \end{cases}$

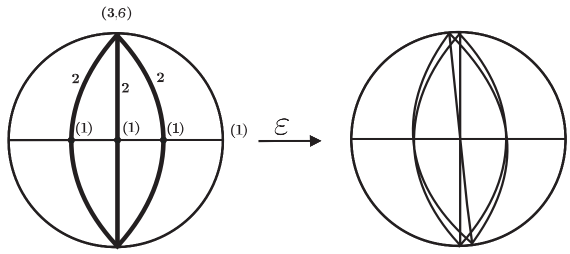

Fig. 4.11. Perturbation of normal form corresponding to the configuration *Config. 8.39*

Config. 8.40: $\begin{cases} \dot{x} = (x-1)(v^2+x^2), \\ \dot{y} = (v^2-2x+3x^2)y; \end{cases}$

Invariant lines : $L_{1,2} = x-1,\ L_{3,4} = x-vi,\ L_{5,6} = x+vi,\ L_7 = y;$

We remark that these systems are obtained from (4.38) due to the transformation $(x, y, t) \mapsto \big((x-1)/2, y, 4t\big)$ and changing u by v.

Config. 8.40$_\varepsilon$:
$$\begin{cases} \dot{x} = (x-1)(v^2 + x^2), \\ \dot{y} = y[v^2 - 2x + 3x^2 - 2\varepsilon(y - 3xy - 2y^2\varepsilon)]; \end{cases}$$

Invariant lines:
$$\begin{cases} L_1 = x - 1,\ L_2 = x - 1 + 2y\varepsilon,\ L_3 = x - vi,\ L_4 = x - vi + 2y\varepsilon, \\ L_5 = x + vi,\ L_6 = x + vi + 2y\varepsilon,\ L_7 = y. \end{cases}$$

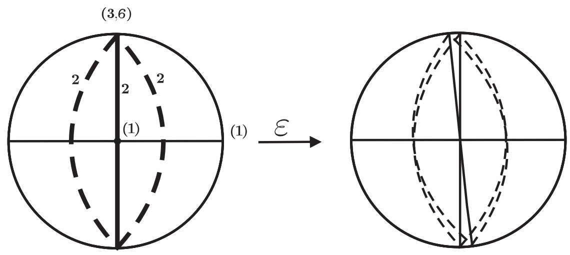

Fig. 4.12. Perturbation of normal form corresponding to the configuration *Config. 8.40*

Config. 8.41 – 8.43 :
$$\begin{cases} \dot{x} = x(r + x^2), \\ \dot{y} = 1 + ry + 3x^2y; \end{cases}$$

Invariant lines: $\quad L_{1,2,3} = x,\ L_{4,5} = x + \sqrt{-r},\ L_{6,7} = x - \sqrt{-r};$

Config. 8.41$_\varepsilon$ *– 8.43*$_\varepsilon$:
$$\begin{cases} \dot{x} = rx + x^3 - \varepsilon + x\varepsilon, \\ \dot{y} = 1 + ry + 3x^2y + \varepsilon(y + 6xy^2 + 4y^3\varepsilon); \end{cases}$$

Invariant lines:
$$L_1 = x + y\varepsilon,\ L_{2,4,6} = rx + x^3 - \varepsilon + x\varepsilon,$$
$$L_{3,5,7} = rx + x^3 + \varepsilon(1 + x + 2ry + 6x^2y + 2y\varepsilon + 12xy^2\varepsilon + 8y^3\varepsilon^2).$$

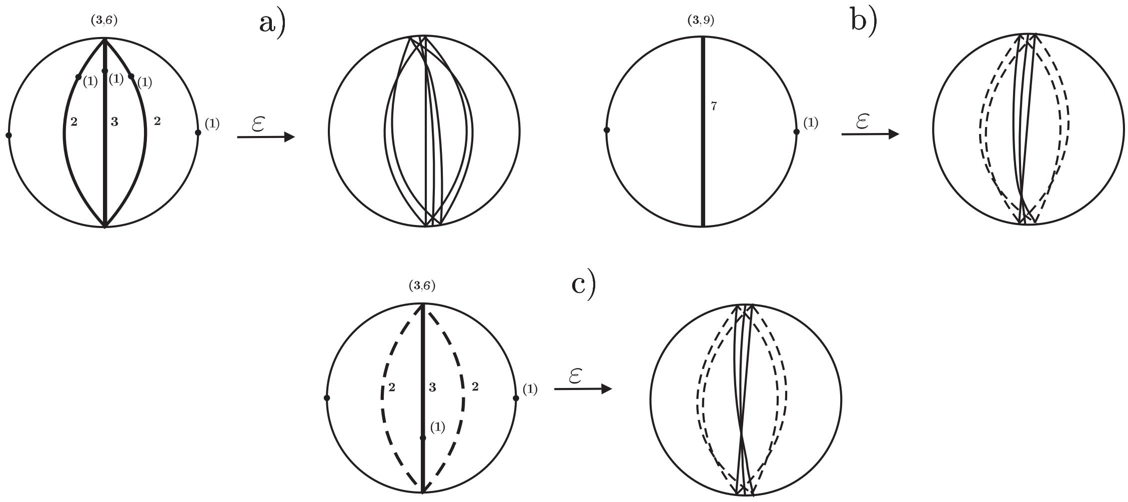

Fig. 4.13. Perturbations of normal forms corresponding to the configurations: a) *Config. 8.41*; b) *Config. 8.42*; c) *Config. 8.43*

Config. 8.44 – 8.46 :
$$\begin{cases}\dot{x} = x(1+x)[u+2+(u+1)x],\\ \dot{y} = y[u+2+(3+2u)x+ux^2];\end{cases}$$

Invariant lines: $L_{1,2,3} = x,\ L_{4,5} = x+1,\ L_6 = (u+1)x+(u+2),\ L_7 = y;$

Config. 8.44$_\varepsilon$ – 8.46$_\varepsilon$:
$$\begin{cases}\dot{x} = x(1+x)[u+2+(u+1)x],\\ \dot{y} = y[(2+u)+(3+2u)x+ux^2-(3+u)xy\varepsilon-(2+u)y^2\varepsilon^2];\end{cases}$$

Invariant lines: $L_1 = x,\ L_2 = x+y\varepsilon,\ L_3 = x+2y\varepsilon+uy\varepsilon,\ L_4 = x+1,$
$L_5 = 1+x+y\varepsilon,\ L_6 = (u+1)x+(u+2),\ L_7 = y.$

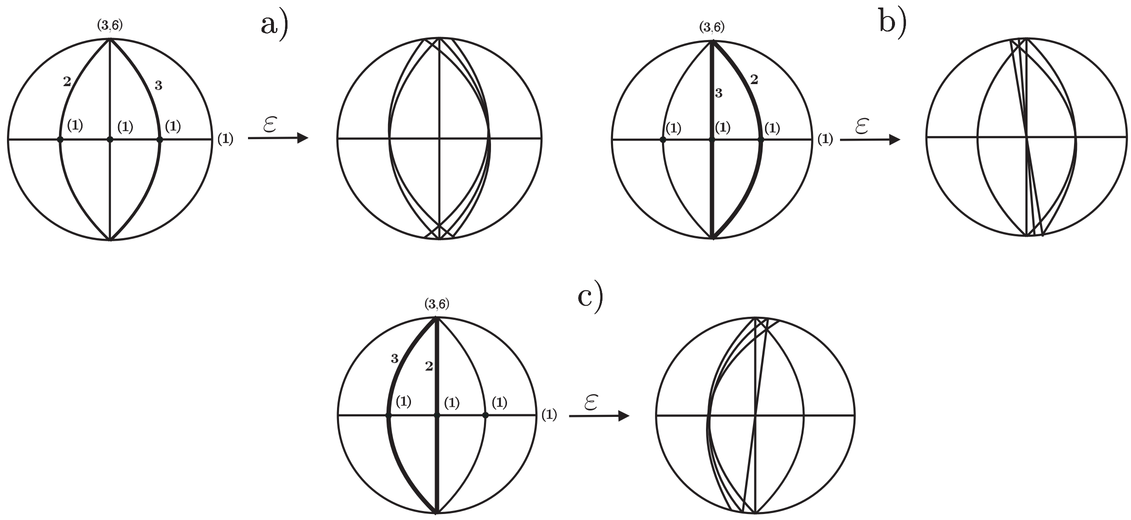

Fig. 4.14. Perturbations of normal forms corresponding to the configurations: a) *Config. 8.44*; b) *Config. 8.45*; c) *Config. 8.46*

Config. 8.47:
$$\begin{cases}\dot{x} = x(x+1),\\ \dot{y} = y(1+x-x^2);\end{cases}$$

Invariant lines: $L_{1,2,3} = x,\ L_{4,5} = x+1,\ L_6 = y,\ L_7 : Z = 0;$

Config. 8.47$_\varepsilon$:
$$\begin{cases}\dot{x} = x(1+x-2x\varepsilon)(1-\varepsilon-x\varepsilon+2x\varepsilon^2)/(2\varepsilon-1)^2,\\ \dot{y} = xy+xy^2(\varepsilon-2)\varepsilon-x^2y(1+\varepsilon)-y(\varepsilon-1)/(2\varepsilon-1)^2+y^3(\varepsilon-1)\varepsilon^2;\end{cases}$$

Invariant lines: $L_1 = x,\ L_2 = x+y\varepsilon,\ L_3 = x+y\varepsilon-y\varepsilon^2,\ L_4 = 2x\varepsilon-y\varepsilon+2y\varepsilon^2-1-x,$
$L_5 = x+1,\ L_6 = y,\ L_7 = 1-\varepsilon-x\varepsilon+2x\varepsilon^2.$

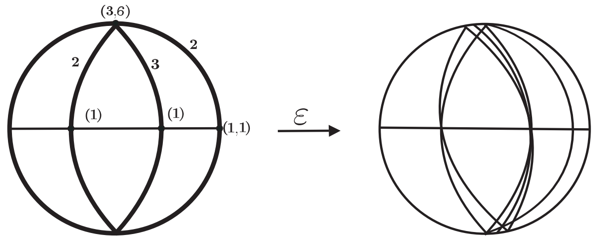

Fig. 4.15. Perturbation of normal form corresponding to the configuration *Config. 8.47*

4.4. One new class of cubic systems with maximum number of invariant lines

In this subsection we show that a new class of cubic systems belongs to $\mathbb{CSL}_9$ is omitted in the classification given by J. Llibre and N. Vulpe (see [16]) in [83].

Indeed, we consider the family of cubic systems with 8 invariant lines (including the line at infinity and including multiplicities) earlier constructed in this chapter

$$\dot{x} = x(r + 2x + x^2), \quad \dot{y} = y(r + 2x), \quad 0 \neq r \in \mathbb{R}, \tag{4.51}$$

which depends on one parameter. These systems have the invariant affine lines: $x = 0$ (triple), $y = 0$, $x^2 + 2x + r = 0$ (simple real or complex or real double) and the line at infinity ($Z = 0$), which is double. We detected that in the case $r = 8/9$ the obtained system

$$\dot{x} = x(2 + 3x)(4 + 3x)/9, \quad \dot{y} = 2(4 + 9x)y/9 \tag{4.52}$$

possesses invariant lines of total multiplicity 9, and namely: $x = 0$ (triple), $x = -2/3$ (double), $x = -4/3$ and $y = 0$ (both simple) and the line at infinity ($Z = 0$, double).

To prove this it is sufficient to present the following corresponding perturbed systems

$$\dot{x} = x(2 + 3x)(4 + 3x)/9, \quad \dot{y} = 2y(1 + \varepsilon y)(4 + 9x - 4\varepsilon y)/9,$$

which possess the following 8 invariant affine lines: $x = 0$, $y = 0$, $x = -2/3$, $x = -4/3$, $3x - 4\varepsilon y = 0$, $3x - 2\varepsilon y = 0$, $1 + \varepsilon y = 0$, $3x - 2\varepsilon y + 2 = 0$. Thus system (4.52) indeed possesses invariant lines of total multiplicity 9 (including the infinite one).

On the other hand in [83] nine classes of cubic systems with two infinite singularities determined by one simple and one triple factors of C_3 are given and their corresponding configurations are presented in Figures 14–22.

Considering the configuration of invariant lines of system (4.52) given in Fig. 5 we observe that this configuration is different from configurations given in Figures 14–22 [83].

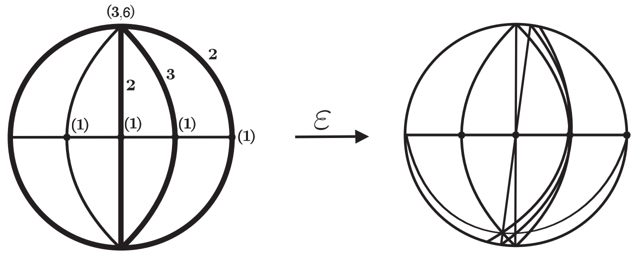

Fig. 4.16. The configuration of invariant lines corresponding to system (4.52) **and its perturbation**

4.5. Conclusions on Chapter 4

We observe that the class of cubic systems in $\mathbb{CSL}_8$ possessing two distinct infinite singularities examined in Chapter 4 give us 25 distinct configurations of invariant straight lines, whereas other three classes in $\mathbb{CSL}_8$ all together could have 26 configurations of invariant straight lines. Moreover the class of systems under the discussion is also interesting because of: 1) it presents a configuration (*Config.8.42*) possessing one affine invariant line with maximum possible multiplicity for cubic systems (i.e. 7) and 2) it was helpful to detect a new class of cubic systems possessing a configuration with invariant lines of total multiplicity nine which was omitted in the classification given by J.Llibre and N.Vulpe [83].

Finally we remark that the configuration given by *Config.8.42* was also detected by Şubă and Vacaraş in [129]. But it is important to underline that in contrast with [129] in Chapter 4 necessary and sufficient conditions for the realization of the configuration given by *Config.8.42* were determined.

The reaserch presented in this Chapter were published in [16, 26, 29].

GENERAL CONCLUSIONS AND RECOMMENDATIONS

The Thesis is devoted to the problem of classifying a family of cubic systems which possess invariant straight lines according to the configurations of these lines. This problem in qualitative study of differential equations, which is very hard even in the simplest case of quadratic differential systems, is partly motivated by *the problem of topologically classifying all phase portraits* of polynomial cubic systems.

As a general observation we note that cubic differential systems are harder to study than quadratic differential systems because of phenomena which could occur in this class but which never occur in the quadratic family. For example the joint presence of limit cycles and singularities which are centers is a phenomenon which occurs in cubic differential systems but does not occur in the quadratic family. Furthermore cubic systems form a family depending on 20 parameters while the class of quadratic differential systems depends on only 12 parameters.

Here we consider the family $\mathbb{CSL}_8$ of cubic systems with invariant lines of total multiplicity eight (including the line at infinity). The study of these systems was based on some concepts, such as the invariant straight line, the multiplicity of lines (of finite/infinite singular points) and the configuration of invariant lines.

An other aspect of the practical and theoretical values of the work is that having all canonical forms of systems in $\mathbb{CSL}_8$, constructed in the current Thesis, *the problem of integrability* of such systems could be solved. Of course we realize that, at the first glance, the class $\mathbb{CSL}_8$ is a very specific one, moreover the cases of integrable systems are rare, but as Arnold said in [1, p.405] "...these integrable cases allow us to collect a large amount of information about the motion in more important systems...".

The main scientific problem which is solved in this Thesis consists in classifying the whole family of cubic differential systems possessing invariant lines of total multiplicity eight according to configurations of these lines; this classification is very helpful for obtaining the complete topological classification of this family and is useful for the study of integrability of this systems.

Novelty and scientific originality of the work consists in the fact that for the first time there are constructed all the possible configurations of invariant lines for systems in $\mathbb{CSL}_8$ and the obtained results are reflected in [8-26]. This set of configurations contains as particular cases all the configurations detected by other authors in special cases of systems in $\mathbb{CSL}_8$ (see [86], [127,128]). In fact, this work is a continuation of [83] where the cubic systems with the maximum number of invariant lines (i.e. 9) were studied and where 23 configurations of invariant lines are detected. However a new class of cubic systems possessing invariant lines of total multiplicity nine which completes the classification given by Llibre and Vulpe in [83] was detected in this Thesis. Thus, we have obtained 51 different configurations of invariant lines. More exactly, we have detected 17 (respectively 5; 25; 4) distinct configurations for the subfamily of systems possessing four (respectively three; two; one) distinct infinite singularities, real or/and complex. At the same time we have constructed 33 canonical forms of the systems possessing invariant lines of total multiplicity eight. We point out that 19 of these canonical forms are one-parameter families, whereas 14 of them are systems with constant coefficients. One more result obtained in the Thesis is the construction of perturbed canonical forms which prove that taking jointly the invariant lines they produced a maximum of eight distinct invariant lines.

The benefits of our elaborations are that this classification, which is taken modulo the action of the group of the real affine transformations and time rescaling, is given in terms of invariant polynomials. The author of the current Thesis has constructed 52 new invariant polynomials besides 20, which were constructed in [83]. It is worth to mention that it was made a great work, because the process of the construction of invariant polynomials takes time and it is pretty difficult. These algebraic invariants and comitants allow one to verify for any given real non-degenerate cubic system with non-degenerate infinity whether or not it has invariant lines of total multiplicity eight, and to specify its configuration of lines endowed with their corresponding real singularities of this system. The important fact is that the calculations can be implemented on computer.

In addition to complex investigations on the research problem, the contribution of the author is materialized by the following **main conclusions** of the Thesis:

1) In this Thesis we studied a whole family of cubic systems, i.e those possessing invariant straight lines of total multiplicity 8. We show that for this class, which we denote by $\mathbb{CSL}_8$, several normal forms are needed each one depending of at most one parameter. To obtain *global results* we used the invariant theory of polynomial differential systems as developed by Sibirschi and his school. *This method allowed us to glue in a unique global diagram, the bifurcation diagrams of the several normal forms needed in the study of this family.*

2) The global result mentioned above is a bifurcation diagram in the 20 dimensional parameter space of cubic differential systems. *This gives us an algorithm to decide for each cubic system whether it belongs to this family or not and in case it belongs to this family, it allows us to effectively compute its specific configuration of invariant lines. We proved that this family possesses **51** possible configurations of invariant straight lines.*

3) The Thesis also led us to obtain the following *new global results*:

(i) a system in $\mathbb{CSL}_8$ must have at infinity at least one real singularity;

(ii) a cubic system with real infinite singularities defined by two double factors of the invariant polynomial $C_3(x,y) = yp_3(x,y) - xQ_3(x,y)$ could not belong to the class $\mathbb{CSL}_8$ and this is the unique exception of cubic systems with real infinite singularities.

The scientific results obtained could be used for a deeper investigation of cubic systems possessing invariant straight lines of total multiplicity eight (including the line at infinity). We propose the following **recommendations:**

(a) the configurations of invariant lines detected, and canonical forms could be used for a complete topological classification of cubic systems in this class;

(b) the canonical forms of cubic systems in $\mathbb{CSL}_8$ constructed can serve as a basis for determining of the first integrals of such systems;

(c) to use the polynomial invariant we constructed for further investigations of cubic systems with invariant lines of total multiplicity less than 8;

(d) to apply the scientific results obtained, in the study of some mathematical models which are described by polynomial differential systems and which are related with some problems in physics, chemistry, medicine, etc.

(e) these investigations could serve as a support for teaching courses in higher education.

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RESPONSIBILITY DECLARATION

I, hereby, confirm that the scientific results presented in the PhD Thesis refer to my investigations. I understand, that in the contrary case I have to face the consequences.

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2. BUJAC C. One subfamily of cubic systems with invariant lines of total multiplicity eight and with two distinct real infinite singularities. In: Bul. Acad. Ştiinţe Repub. Mold. Mat., 2015, no.1(77), p. 48-86.
3. BUJAC C., VULPE N. Cubic differential systems with invariant straight lines of total multiplicity eight and four distinct infinite singularities. In: J. Math. Anal. Appl. 423 (2015), p. 1025-1080 .
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