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ANALYTIC GEOMETRY OF HOMOGENEOUS SPACES

111.04 GEOMETRY AND TOPOLOGY

Synopsis of Doctor Thesis in Mathematical Sciences

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THESIS CONCEPTUAL REFERENCE POINTS

Actuality of theme. The history of non–Euclidean geometry started less than 200 years ago. In 1829, Nikolai Lobachevsky, and independently of him, in 1831, János Bolyai published the axiomatically developed theory where non–Euclidean geometry, presently known as hyperbolic, was considered. Arthur Cayley introduced in 1859 different ways to define space metric. Felix Klein in 1871, using Cayley's metric, constructed models of 9 plane geometries. Klein proposed "Erlangen Program" [7], a new solution to the problem how to classify and characterize geometries on the basis of projective geometry and group theory. Euclidean, hyperbolic and elliptic geometries, known until then, found their place in Klein's classification. In 1892, Hendrik Lorentz presented the transformations of space–time. In 1905 Henri Poincaré observed that the Lorentz transformation is actually a rotation by hyperbolic angle. The group of Lorentzian transformations is one of the most important Lie group. Due to Lie group — Lie algebra correspondence, it is possible to study geometric objects using algebraic language. The classification of homogeneous spaces based on the algebraic classification of their Lie groups is the the attention of researchers in these days. The Lorentz's idea was taken in the thesis for generalization of the notion of rotation for any homogeneous space.

Albert Einstein in 1905 developes the idea of the geometrization of physics, proposed by Lorentz, in special relativity theory which was further developed by Hermann Minkowski in 1907. Minkowski geometry found its place in Klein's classification. Also the calssical kinematics of Galilei–Newton found its place in Klein's classification, if it is given the geometric interpretation. Different studies with applications in various domains of the physical mathematics were undertook by Hermann Weyl (in 1913), Élie Cartan, George Vranceanu, Finsler, Radu Miron and others (1960 — 1970's years). When Willem de Sitter proposed the cosmologic model of early evolution of universe in 1917, it was observed that also spaces De Sitter and Anti de Sitter have their reserved places in Klein's classification. Last achievements in theoretical physics (namely, in the String Theory, Green, Schwartz, 1970 — 1980's) generated new homogeneous spaces which leads to necessity of developing new methods of study. The methods of linear algebra provide the universal instruments of research. It follows the study of different mathematical structures by linear methods represents an actual direction of research, which is important for domains of mathematics as well as for its applications.

Current situation in the research domain. The differential geometry of homogeneous spaces was elaborated around 1923 by Élie Cartan and further developed by Ehresmann Ch. similarly to what was previously done by Riemann for constructions made by Euclid, Lobachevsky and Bolyai. Riemann introduced the term "metric", Cartan introduced what is now known as "Cartan connection". In the following several decades, the differential geometry bacame the main tool to study homogeneous spaces. The new approach of the differential geometry related with homogeneous spaces was contribued by Buseman G., Bachmann F., Efimov N.V, Hjelmslev J., Nash L.F., Kallenberg G.W., Borisov Yu.V., Borisenko A.A., Milka A.D., Verner A.L., Schwartz J. T., Naoum A., Roitberg J., Klingenberg R., Karzel H., Struve H., Struve R. etc. (see [1, 4, 6, 8, 9, 21, 22, 23, 24]).

The research domain of homogeneous spaces has connections with many compartments of contemporary geometry, from the most recent contributions we mention the monography of authors Bourguignon

J. P., Hijazi O., Milhorat J. L., Moroianu A., Moroianu S. [3]. An essential contribution in differential geometry of spaces Finsler, Lagrange, Hamilton and their generalizations, which are closely related with homogeneous geometry, belongs to Miron R. and Anastasiei M. [11], Udriște C. and Balan V. [2].

Important results for some homogeneous spaces of different dimensions were obtained also in the domain of discrete geometry. We mention the works elaborated by Zamorzaev A. [20] on development of the theory of symmetry of homogeneous (Euclidean, pseudo–Euclidean, Minkowski), results in domain of discrete hyperbolic geometry developed by Macarov V. [10] and in domain of hyperbolic manifolds obtained by Guțul I. [5] and Damian F. [25].

In metric geometry of homogeneous spaces we mention some results upon which the present work is based. Rosenfeld B., Yaglom I., Evghenia Yasinskaya E. [27, 31] have contribued to the classification and development of the methods of study of two–dimensional homogeneous spaces. Romakina L. described several two–dimensional homogeneous spaces [28, 29, 30]. The author works can be consulted in publications [12, 13, 14, 15, 16, 17, 18, 19, 26].

Goals and objectives. The goal of the research is to provide a toolchain that can be used to study any homogeneous space by means of analytic geometry and linear algebra. In order to achieve this goal, the following objectives have to be fulfilled:

- Introduction of a new concept of the space signature;
- Construction of homogeneous space based on the concept of signature;
- Construction of a model of homogeneous space for each given signature;
- Expression of the measurement of different geometric quantities via signature, which reflects their role in analytic geometry of homogeneous spaces;
- Finding different applications of the analytic geometry of homogeneous spaces.

Research methodology. The theory uses methods and language of linear algebra to study non–linear spaces. As subject of study, the *model* of homogeneous spaces is constructed. The methods of *linear algebra* are used. In order to see that the model is adequately constructed, its relevant to research properties are verified.

Thesis investigates the notion of homogeneous space and establishes what kind of properties are to be expected. This is achieved by *generalization* of axioms of known geometries. Then, using *duality* method new properties are formulated, dual to known ones.

An important method used in research is *computer modelling*, realized as software project GeomSpace. This project is based on theory results and also is used to test conjuctures, to provide counterexamples and to obtain new results.

Scientific innovation. The main innovation of elaborated theory is space *parameterization* by the introduction of space signature. This parameterization allows studying different homogeneous spaces in one global framework. When parameters are used as variables in definitions, axioms, equations, theorems, proofs, all these have exactly the same form that describes the reality of all homogeneous spaces simultaneously. When it is necessary to describe some space particularities or to see the difference between two concrete spaces, concrete values can be put in parameters of each definition, axiom, equation, theorem and proof.

The parameterized approach has many advantages. Firstly, because of the large number of homogeneous spaces, it is next to impossible to describe each space with its geometry one by one. A uniform approach allows to describe any concrete space with all its particularities or to compare two spaces. Secondly, some results are more easy to obtain for non–linear spaces, and then they can be generalized to linear ones. Other results are more easy obtained for linear spaces, and then they can be extended to non–linear ones. A uniform approach drastically simplifies such extensions and generalizations.

Important scientific problem solved is the research of the homogeneous spaces via linear methods applying the concept of signature.

Theoretical value of the thesis. This work provides one possible universal terminology across different spaces that facilitates comparison of different spaces properties.

Practical value of the work. Although the thesis is focused on analytic geometry is has value for differential geometry. The modern differential geometry methods are constructed based on analytic geometry of Euclidean space. This fact has two consequences. First, this approach is useless to describe some non–Euclidean space property that is missing from Euclidean space or to compare such properties of two non–Euclidean spaces. Second, differential geometry has some restrictions in description of homogeneous spaces that can't be approximated by Euclidean space in any point. The described theory may be used as toolchain for more universal differential geometry.

Obtained scientific results to be defended:

- Classification of homogeneous spaces based on new form of signature, introduced in this work;
- Elaboration of the universal form of trigonometric equations, common for all homogeneous spaces;
- Introduction of the group of generalized orthogonal matrices and its study in connection to the motion group of homogeneous space;
- Adaptation of the routines and algorithms of linear algebra, that operate on vectors or on vector families, to homogeneous spaces;
- Introduction of decomposition vectors for limit vectors (isotropic vectors) and study of limit vectors with aim of their decomposition vectors;
- Definition and study of volumes in a homogeneous space by corresponding volumes in its metaspase;
- Theorem on isomorphism of the crystallographic groups of dual homogeneous spaces.

Implementation of scientific results. Because of theory simplicity, it can be used as facultative course in lyceum or university in order to develop the geometric intuition.

Another implementation is software project Geomepace (http://sourceforge.net/ projects/geomspace/) having the goal to provide an interactive geometry environment for homogeneous spaces.

Approval of work. Different aspects of the theory were presented at the following scientific events:

2009 Alba Iulia, România — International Conference on Theory and Applications in Mathematics and Informatics,

2010 Moscow, Russia — International conference "Metric geometry of surfaces and polyhedra", dedicated to 100th anniversary of N. V. Efimov,

2010 Moscow, Russia —The International Conference "Geometry, Topology, Algebra and Number Theory, Applications" dedicated to the 120th anniversary of B. N. Delone,

2014 Chişinău, Moldova — The Third Conference of Mathematical Society of Moldova, IMCS-50,

2015 Tula, Russia — International conference "Algebra, Number Theory and Discrete Geometry: Modern Problems and Applications", dedicated to 85th anniversary of professor S. S. Ryshkov.

2015 Iași, România — The 8th Congress of Romanian Mathematicians,

2016 Chişinău, Moldova — International Conference Mathematics & Information Technologies: Research and Education, MITRE — 2016.

Research papers: The thesis results were published in 9 works: [12, 13, 14, 15, 16, 17, 18, 19, 26], 5 articles and 4 communications on national and international conferences: 2 articles were published in 2 peer–reviewed journals (one of category A another of category B+). All publications are single author.

Keywords: homogeneous space, Riemannian space, Klein geometry, projective metric, analytic geometry.

Structure of the thess. The thesis is written in English and consists of: introduction, three chapters, general conclusions and recommandations, appendix, 210 bibliography titles, 140 pages of main text, annotations in Romanian, Russian and English.

THESIS CONTENTS

In the **Foreword**, the actuality of theme is argued, also there are specified goals and objectives and methodology of the research. The short thesis contents is given.

The Chapter 1 **“Analysis of situation in domain of homogeneous spaces”** introduces the algebraic structures used in the research, presents an overview of current situation in the thesis area, analyzes the advantages and limits of the axiomatic and the modelling scientific approaches.

Definition 1.2.1 (Homogeneous space). Let X be a non–empty set and G a group that acts on X. The structure of X is $\tau : G \to Aut(X)$. A pair (X, τ) is called a *homogeneous space*, if:

- τ is a homomorphism, that is for each $g \in G$ the mapping $\tau(g)$ is structure preserving;
- $\tau(G)$ acts transitively on X.

The duality principle is described. The main thesis results are presented.

In the Chapter 2 **“Analytic Geometry”**, the model of homogeneous space is constructed and its metric properties are studied. In section 2.1 **“Definition and type of generalized rotations”** the basic properties which are expected in any model of homogeneous space are introduced.

Definition 2.1.2 (Generalized trigonometric functions). Let $k \in \{-1, 0, 1\}$ and:

$$C(\varphi) = \sum_{n=0}^{\infty}(-k)^n \frac{\varphi^{2n}}{(2n)!}, \quad S(\varphi) = \sum_{n=0}^{\infty}(-k)^n \frac{\varphi^{2n+1}}{(2n+1)!}, \quad T(\varphi) = \frac{S(\varphi)}{C(\varphi)}.$$

We call functions $C(\varphi)$, $S(\varphi)$ and $T(\varphi)$ *generalized cosine*, *sine* and *tangent* respectively.

In the section 2.2 **“Homogeneous Space Model”**, the linear and parameterized model of homogeneous spaces is constructed.

Definition 2.2.1 (Main space rotations). We call *main space rotations* the transformations $\mathfrak{R}_i(\varphi)$, $i = \overline{1,n}$, expressed by square matrices of the order $n+1$, having the elements:

$$r_{i-1\,i-1} = r_{ii} = C_i(\varphi), \quad r_{i\,i-1} = S_i(\varphi), \quad r_{i-1\,i} = -k_i S_i(\varphi), \quad r_{pq} = \delta_{pq}, p, q \neq i, j.$$

Consider vector space $\mathbb{R}^{n+1}$ and a collection of numbers $k_1, k_2, ...k_n \in \{-1, 0, 1\}$, each being the type of one main rotation. Let $C_i(x) = C(x, k_i)$, $S_i(x) = S(x, k_i)$ and $T_i(x) = \frac{S_i(x)}{C_i(x)}$.

Definition 2.2.2 (Meta product of vectors). We define *meta product of vectors* $\odot$ as

$$x \odot y = \sum_{i=0}^{n} K_i x_i y_i, \text{ where } K_0 \equiv 1, \quad K_m = \prod_{i=1}^{m} k_i, \quad m = \overline{1,n}.$$

Definition 2.2.3 (Homogeneous space model, signature, metaspace). Define *homogeneous space* $\mathbb{B}^n$ as $B^n \subset \mathbb{R}^{n+1}$. We call the collection of numbers $\{k_1, ..., k_n\}$, each being the type of corresponding main rotation, the *signature* of homogeneous space. We call $\mathbb{R}^{n+1}$ with meta product of vectors $\odot$ a *metaspace*. The vector $e = (1 : 0 : ... : 0)$ is called the *origin* $E = (1 : 0 : ... : 0)$.

Remark. Two opposite vectors $x, -x \in \mathbb{R}^{n+1}$ of the model represent the same vector of a homogeneous space.

Definition 2.2.4 (Motion, line, plane). Define *motions* of $\mathbb{B}^n$ as all possible transformations that are composed of finite products of main rotations. Define *lines* as the images $\mathfrak{M}(\mathbb{B}^1)$ under all possible motions $\mathfrak{M} : \mathbb{B}^n \to \mathbb{B}^n$. Similarly, define *m-dimensional planes* as images $\mathfrak{M}(\mathbb{B}^m)$ under all possible motions $\mathfrak{M}$, $m = \overline{0, n-1}$.

Definition 2.2.6 (Distance, angle). We say, the *distance* between a point $A \in \mathbb{B}^1 \subset \mathbb{B}^n$ and the origin E is φ, if $A = \mathfrak{R}_1(\varphi)E$. We say, the *one–dimensional (plane) angle* between $\mathbb{B}^1$ and some one–dimensional line $\mathbb{B}^0 \subset \mathbb{B}'^1 \subset \mathbb{B}^2$ is equal to φ, if $\mathbb{B}'^1 = \mathfrak{R}_2(\varphi)\mathbb{B}^1$. Similarly, define $(m+1)$*–dimensional angle* φ between $\mathbb{B}^m$ and m-dimensional plane $\mathbb{B}^{m-1} \subset \mathbb{B}'^m \subset \mathbb{B}^{m+1}$, if $\mathbb{B}'^m = \mathfrak{R}_{m+1}(\varphi)\mathbb{B}^m, \quad \forall m = \overline{0, m-1}$.

In the section 2.3 **"Relations in Triangle"**, we deduce equalities and inequalities of the triangle and equalities of the right triangle.

Table 2.1: Triangle equalities

$\dfrac{S_1(a)}{S_2(\alpha)} = \dfrac{S_1(b)}{S_2(\beta')} = \dfrac{S_1(c)}{S_2(\gamma)}$
$C_1(a) = C_1(b)C_1(c) + k_1 S_1(b)S_1(c)C_2(\alpha)$
$C_1(b) = C_1(a)C_1(c) - k_1 S_1(a)S_1(c)C_2(\beta')$
$C_1(c) = C_1(a)C_1(b) + k_1 S_1(a)S_1(b)C_2(\gamma)$
$C_2(\alpha) = C_2(\beta')C_2(\gamma) + k_2 S_2(\beta')S_2(\gamma)C_1(a)$
$C_2(\beta') = C_2(\alpha)C_2(\gamma) - k_2 S_2(\alpha)S_2(\gamma)C_1(b)$
$C_2(\gamma) = C_2(\alpha)C_2(\beta') + k_2 S_2(\alpha)S_2(\beta')C_1(c)$

Proposition 2.3.1. The longest edge is opposed to the largest angle and the shortest edge is opposed to the smallest angle.

The triangle inequality, that plays an important role in the definition of metrics, in homogeneous spaces generalizes the well–known Euclidean one. This fact is reflected in the following proposition.

Proposition 2.3.2. The shortest edge a and the longest edge b of the triangle are:

$$a \begin{cases} > b - c, & k_2 = 1; \\ = b - c, & k_2 = 0; \\ < b - c, & k_2 = -1. \end{cases} \qquad b \begin{cases} < a + c, & k_2 = 1; \\ = a + c, & k_2 = 0; \\ > a + c, & k_2 = -1. \end{cases}$$

Internal angle α and external angle β' are:

$$\alpha \begin{cases} > \beta' - \gamma, & k_1 = 1; \\ = \beta' - \gamma, & k_1 = 0; \\ < \beta' - \gamma, & k_1 = -1. \end{cases} \qquad \beta' \begin{cases} < \alpha + \gamma, & k_1 = 1; \\ = \alpha + \gamma, & k_1 = 0; \\ > \alpha + \gamma, & k_1 = -1. \end{cases}$$

Table 2.2: Right triangle equalities

$T_1(b) = T_1(c)C_2(\alpha)$	$T_{12}(a) = T_1(c)S_2(\beta')$
$S_{12}(a) = S_1(c)S_2(\alpha)$	$S_1(b) = S_1(c)C_2(\beta')$
$T_{12}(a) = S_1(b)T_2(\alpha)$	$S_{12}(a) = T_1(b)T_2(\beta')$
$C_2(\alpha) = C_{12}(a)C_2(\beta')$	$S_2(\beta') = C_1(b)S_2(\alpha)$
$C_1(c) = C_{12}(a)C_1(b)$	$T_2(\beta') = C_1(c)T_2(\alpha)$

The presence of universal form of the triangle equalities can be generalized to any straight–lined figure.

Proposition 2.3.3. For any straight–lined figure Ω in homogeneous space $\mathbb{B}^n$, if there exists equation:

$$F(p_1, ..., p_n) = 0,$$

that relates elements $p_1, ..., p_n$ of this figure, then it is possible to find its form:

$$H(Tr(p_1), ..., Tr(p_n)) = 0,$$

which is expressed through functions having the properties:

- Function H is algebraic and doesn't depend on space $\mathbb{B}^n$;
- Function $Tr(p_i)$ is any one from these three: $C_i(p_i)$, $\sqrt{k_i}S_i(p_i)$, $\sqrt{k_i}T_i(p_i)$, which depends only on type k_i of its argument p_i.

Corollary 2.3.3. The properties described by equation H are properties of figure Ω and don't depend on the space, the properties described by function Tr are properties of space $\mathbb{B}^n$ and don't depend on figure.

In the section 2.4 **"Motion"**, the generalized orthogonal matrices are introduced and the theory of isometry of homogeneou space is studied by means of generalized orthogonal matrices.

Let us generalize the meta product of vectors. For this purpose, we generalize the type K_{ij}:

$$K_{ij} = 1,\ i = j; \quad \prod_{p=i+1}^{j} k_p,\ i < j; \quad \frac{1}{K_{ji}},\ i > j.$$

Definition 2.4.2 (i-th product). Define *i-th product*, $i = \overline{0, n}$ of vectors x and y as follows:

$$x \odot_i y = \sum_{j=0}^{n} K_{ij} x_j y_j. \quad (1.1)$$

Definition 2.4.3 (Vector index). We say, the number i, $0 \le i \le n$ is *index of vector* $x \in \mathbb{R}^{n+1}$, if $x \odot_i x > 0$. In this case the notation is x^i.

Theorem 2.4.3. Vector index doesn't depend on space basis choice.

Definition 2.4.4 (Natural product). Define *natural* product of vectors x^i and y^j, the one that has index $k = \min(i, j)$: $x^i \odot y^j = x^i \odot_{\min(i,j)} y^j$.

Definition 2.4.5 (Normalized and orthogonal vectors, generalized orthogonal matrix). We call vector x^i in $\mathbb{R}^{n+1}$ with given signature, *normalized*, if $x^i \odot x^i = 1$. We call vectors x^i, y^j *orthogonal*, if $x^i \odot y^j = 0$. Define square matrix M of size $n + 1$ *generalized orthogonal* if all its columns m^j have index j, are normalized and any two columns are orthogonal. Further, for the generalized orthogonal matrices acting in metaspace the shorter term is also used: *GM-orthogonal.*

Theorem 2.4.11. GM-orthogonal matrices form a group. The isometry group (group of motions) of space is a subgroup of this group.

Algorithm 2.1 (GM-orthogonal matrix decomposition in product of rotations).

1. For rows r from n to 1 do:

   (a) Divide elements of the row x_{ri}, $i = \overline{0, r}$ in three categories: having the type K_{ri} equal to 1, 0 and -1. We will multiply from right X by $\mathfrak{R}_{ir}(\varphi)$, $i = \overline{0, r}$ so that in r-th row one element of category 1 remains and one element of category -1 remains, both different from 0. For elements x_{ri} and x_{rj} with the same type we can use $\cos\varphi = \frac{x_{ri}}{\sqrt{x_{ri}^2 + x_{rj}^2}}$ and $\sin\varphi = \frac{x_{rj}}{\sqrt{x_{ri}^2 + x_{rj}^2}}$.

   (b) Now have one element of types 1 and -1 different from zero (say they are r-th and p-th), There exists $\varphi \in \mathbb{R}$ so that $\cosh\varphi = \frac{x_{rr}}{\sqrt{x_{rr}^2 - x_{rp}^2}}$ and $\sinh\varphi = \frac{-x_{rp}}{\sqrt{x_{rr}^2 - x_{rp}^2}}$.

   (c) For category 0 parabolic rotations exist. For this, if one such element is in q-th columns, $\varphi = -\frac{x_{rq}}{x_{rr}}$.

   (d) Elliptic, hyperbolic and parabolic rotations transform also the elements from 0th to (r-1)th of the column r into zero.

2. At this phase we can consider the obtained matrix as having the size r instead of $r + 1$ and repeat the process for it. At the end obtain matrix $E = diag(\pm 1)$. We get equality: $X \prod_{j=1}^{q} M_j = E$, so $X = E \prod_{j=q}^{1} M_j^{-1}$.

Definition 2.4.7 (Equivalent, interchangeable and non-interchangeable vectors)**.** We say two vectors with indices i, j are *non-interchangeable*, if there is no homogeneous space isomorphic to given one in which corresponding two vectors have indices j, i. We say vectors are *interchangeable*, if such isomorphic homogeneous space exists. We say vectors are *equivalent*, if space contains a motion that interchanges them (up to sign).

Theorem 2.4.17. Two vectors $e^i, e^j, \quad i < j$ are:

- non-interchangeable, if $K_{ij} = 0$;
- interchangeable, if $K_{ij} = -1$;
- equivalent, if $K_{ij} = 1$.

In section 2.5 *"Lineal"*, different algorithms, which operate on vector families, are deduced.

Definition 2.5.1 (Lineal)**.** We call a *lineal* any intersection of the sphere $\mathbb{B}^n$ and the linear span of a family of vectors of the metaspace.

All planes are lineals. But not all lineals are planes. Obviously, congruent lineals have equal signatures. However, not all lineals with equal signatures are congruent.

Definition 2.5.3 (Projection of vector on lineal)**.** Define v' to be the *projection* of vector v on linal L^m, if $v' \in L^m, v'' = v - v' \perp L^m$. Define v'' to be the *projection* of vector v on orthogonal complement of lineal L^m.

Lemma 2.5.1. The projection v' of vector v on lineal L^m can be computed as:

$$v' = \sum_{i=0}^{m} (v \odot_i l^i) l^i,$$

where vector family $\{l^i\}_{i=\overline{0,m}}$ is an orthonormal basis of L^m.

Consider a vector family $\{x^i\}_{i=\overline{0,m}}$. In order to orthonormalize it, consider the following algorithm:

Algorithm 2.2 (Vector family orthonormalization)**.**

1. Take the vector with lowest index x and normalize it. Add y to family of orthonormal vectors $\{y^i\}$ and remove x from initial vector family $\{x^i\}$.
2. While family $\{x^i\}$ contains at least one vector do.
   (a) Choose from family $\{x^i\}$ the vector with lowest index x and find its projection on the orthogonal complement of $\{y^i\}$.
   (b) Remove vector x from family $\{x^i\}$, and if x' is nonzero, normalize it and add to family $\{y^i\}$.

Consider orthonormal vector family $\{x^i\}_{i=\overline{0,m}} \in \mathbb{B}^n$, $m < n$. The objective of the following algorithm is to complete this family to contain $n + 1$ of vectors so that all vectors to be orthonormal.

Algorithm 2.3 (Orthonormal vector family completion)**.**

1. For each coordinate vector $\{e^j\}_{j=\overline{0,n}}$ find a vector y orthogonal to family $\{x^i\}$. If vector y isn't zero, normalize it and add to family $\{x^i\}$.

It is necessary to find the unique form of lineal determination. In order to find such a basis, the following algorithm can be used:

Algorithm 2.4 (Canonical form of lineal basis)**.**

1. Consider coordinate vector family $\{e^p\}_{p=\overline{0,n}}$ of space $\mathbb{R}^{n+1}$. Start with empty basis of L.
2. Until new basis has $m+1$ elements, find projection e'^p of the next e^p on L^m.
   (a) If e'^p is nonzero, find projection e''^p of e'^p on orthogonal complement to existing basis $\{l^i\}$ of lineal L.
   (b) If e''^p isn't zero, normalize it and add to existing basis $\{l^i\}$ of lineal L.

In the section 2.6 **"Limit Vectors and Lineals"**, the vectors without index, called limit vectors, are studied. In two-dimensional space these vectors are isotropic. The theory of limit lineal is also studied.

Definition 2.6.1 (Limit vector)**.** Define a vector with no index to be a *limit vector.*

Consider a limit vector $x \in \mathbb{B}^n$. Construct vectors $a, b \in \mathbb{B}^n$ so that vector a has all coordinates a_i equal to those of x (coordinates that enter in bilinear form with sign "+" or coefficient 0), the rest equal to 0. Vector b has all coordinates b_j equal to those of x (coordinates that enter in bilinear form with sign "−"), the rest equal to 0.

Definition 2.6.2 (Decomposition vectors of limit vector)**.** We call the vector pair a, b *decomposition vectors* of vector x.

Remark. Definition of limit vector depends on the space basis choice and is not the only possible.

Lemma 2.6.1. Decomposition vector indices do not depend on the space basis choice.

Theorem 2.6.2. (Mean characteistic). Limit vector type is always equal to 0.

Lemma 2.6.3. The value of limit vector measure $x \in \mathbb{B}^n$ is equal to the values of its decomposition vectors measurs, a and b:

$$|x| = |a| = |b|.$$

Remark. The measure of limit vector depends on the space basis choise and *is not* invariant under motions. Thus, it can't be considered a measure in the strict sense.

Since for limit vector x the following equality holds, $x \odot x = 0$, there are situations where one limit vector is orthogonal to some indexed vector or to another limit vector, however the decomposition vectors of limit vectors are not orthogonal to indexed vector or among them. In the following, the procedure of orthogonalization of limit vectors in such situations is presented.

Proposition 2.6.6. We will consider limit vectors orthogonal to indexed vectors or orthogonal among them, if all their decomposition vectors are orthogonal to indexed vectors or are mutually orthogonal.

Definition 2.6.3 (Limit lineal)**.** Define *limit lineal* as a lineal with orthonormal basis that contains limit vectors.

Lemma 2.6.7. If an orthonormal basis of some lineal contains limit vectors, their decomposition vectors indices are free in lineal.

Proposition 2.6.8. Consider composed index of limit vectors as its decomposition vectors index pair. This index doesn't depend on the space basis choice and is free in lineal. Denote $x^{ij} = a^i + b^j$, where vectors a^i, b^j are decomposition vectors of limit vector x^{ij}.

The limit lineal signature can't be deduced from the space signature and basis vector indices.

Algorithm 2.5 (Limit lineal signature)**.**

1. Divide all basis vectors of lineal into equivalence groups so that in each group there exist only equivalent and / or interchangeable vectors, and each vector of one group is non-interchangeable to all vectors of the next group. Instead of limit vectors use corresponding decomposition vector pairs.

2. In each group, as earlier, find the elements of signature as types of motion from one indexed vector to another.

3. In each group add to signature the number 0 as the type of motion from the last indexed vector to the first limit vector, if such vectors exist.

4. In each group add to signature the number 1 as the type of motion between limit vectors, if there exist more than one.

5. Insert the number 0 between groups of non-interchangeable vectors, from the first to the last.

In the section **"Constructions and Calculus"** several computational algorithms are deduced. Homogeneous space $\mathbb{B}^n$ exactly coincides with metaspace $\mathbb{R}^{n+1}$ up to point normalization. The metaspace is a linear vector space. So, many constructions, calculus and algorithms for linear vector spaces are applicable to $\mathbb{B}^n$ space with one mention: the result have always to be normalized.

In the following algorithms, consider lineals A^p, B^q with orthonormal bases $\{a^i\}_{i=\overline{0,p}}$ and $\{b^j\}_{j=\overline{0,q}}$.

Algorithm 2.6 (Lineal difference).

1. Find projections a'^i of vectors a^i to orthogonal complement of B^q.

2. Orthonormalize family $\{a'^i\}$.

Algorithm 2.7 (Lineal sum and intersection).

1. Denote the basis of sum by $\{w^i\}$, of intersection by $\{v^i\}$ and of one more vector family by $\{h^i\}$.

2. Copy all vectors $\{a^i\}$ to $\{w^i\}$ and $\{h^i\}$, fill with zero vectors all free indices of $\{w^i\}$ and $\{h^i\}$, so that each family contains $n+1$ vectors.

3. For each vector b^i do:

   (a) Find vector:

   $$b'^i = b^i - \sum_j (b^i \odot w^j) w^j,$$

   orthogonal to family $\{w^i\}$. Using the same coefficients, calculate:

   $$h = -\sum_j (b^i \odot w^j) h^j.$$

   (b) If b'^i isn't zero, it is linearly independent with $\{w^i\}$. Normalize it:

   $$w = \frac{1}{\sqrt{b' \odot b'}} \cdot b'$$

   and add to $\{w^i\}$. Using the same factor, add to $\{h^i\}$ the vector:

   $$h' = \frac{1}{\sqrt{b' \odot b'}} \cdot h.$$

   (c) If b'^i is zero, orthonormalize h with $\{v^i\}$ and add to it.

4. Exclude zero vectors from family $\{w^i\}$. The family $\{v^i\}$ is the intersection basis and the family $\{w^i\}$ is the sum basis.

Consider a space $\mathbb{B}^n$ with $m+1$ vectors $\{v^i\}_{i=\overline{0,m}}$. Let v^i coordinates be $(v_{0i} : \ldots : v_{ni}), i = \overline{0,m}$. And let vectors v^i be ordered and linearly independent. Compose the matrix V with elements $\{v_{ij}\}, i = \overline{0,n}, j = \overline{0,m}$.

Definition 2.7.2 (Coordinate matrix, state matrix). Define *coordinate matrix* of vector family $\{v^i\}$ to be a rectangular matrix composed of this vector family coordinates. Define *state matrix* of vector family to be a matrix composed of elements $v^i \odot_i v^j$.

Proposition 2.7.10. If the measure between lineals equals to zero or they are orthogonal, then the measure type is, generally speaking, ambiguous.

Algorithm 2.8 (Measure between lineals).

1. Find orthonormal bases $\{a^i\}, \{b^j\}$ of the lineals $A^p - B^q$ and $B^q - A^p$. Consider the number of vectors $\{a^i\}$ doesn't exceed that of $\{b^j\}$. Otherwise interchange the roles of lineals.

2. Decompose vectors $a^i = a'^i + a''^i$, where the vectors $\{a'^i\}$ are projections of vectors $\{a^i\}$ on $B^q - A^p$, and vectors $\{a''^i\}$ on the orthogonal complement to $B^q - A^p$.

3. Find determinants w', w'' of the status matrices W', W'' of vector families $\{a'^i\}, \{a''^i\}$.

   (a) If $w' = 1, w'' = 0$, then $\varphi = 0$, ψ is undefined and k is ambiguous.

   (b) If $w' = 0, w'' = 1$, then φ is undefined, $\psi = 0$ and k is ambiguous.

   (c) If $w' + w'' = 1$, then $\varphi = \tan^{-1}\sqrt{\frac{w''}{w'}}$, $\psi = \tan^{-1}\sqrt{\frac{w'}{w''}}$ and $k = 1$.

   (d) If $w' = 1, w'' \neq 0$, then $\varphi = \sqrt{w''}$, ψ is unmeasurable and $k = 0$.

   (e) If $w' \neq 0, w'' = 1$, then φ is unmeasurable, $\psi = \sqrt{w'}$ and $k = 0$.

   (f) If $w' - w'' = 1$, then $\varphi = \tanh^{-1}\sqrt{\frac{w''}{w'}}$, ψ is unmeasurable and $k = -1$.

   (g) If $w'' - w' = 1$, then φ is unmeasurable, $\psi = \tanh^{-1}\sqrt{\frac{w'}{w''}}$ and $k = -1$.

   (h) If at least one of the two lineals is limit and $w' = w''$, then $\varphi = \psi = \infty$ and $k = -1$.

In the section 2.8 **"Volume"**, the volume in homogeneous space is defined by means of the volume in linear metaspace, and the theory of volumes is studied with concrete results in dimension two.

Proposition 2.8.2. If $F \subset \mathbb{B}^n$ is some figure with volume $v_{\mathbb{B}}$ (in sense of $\mathbb{B}^n$), and $v_{\mathbb{R}}$ is volume (in sense of $\mathbb{R}^{n+1}$) of cone with base in $F \subset \mathbb{B}^n$ and vertex at $O \notin \mathbb{B}^n$ origin of $\mathbb{R}^{n+1}$, then

$$v_{\mathbb{B}} = (n+1)v_{\mathbb{R}}.$$

Corollary 2.8.2. The volume $v_{\mathbb{B}}$ doesn't change under motions.

Consider right triangle $\triangle ABC$ with catheti a, b, hypotenuse c, internal angle α and external angle β'. The area s of triangle is:

$$s = \frac{\alpha - \beta'}{k_1}.$$

This equality is convenient when $k_1 \neq 0$. In all other cases we can use one of the equalities from the Table 2.4.

Table 2.4: Right triangle area

$S(s)=\dfrac{S(a)S(b)}{1+C(a)C(b)}$
$C(s)=\dfrac{C(a)+C(b)}{1+C(a)C(b)}$
$T(s)=\dfrac{S(a)S(b)}{C(a)+C(b)}$

Theorem 2.8.3. Area type k is equal to the product of coordinate vectors types:

$$k = K_1 K_2.$$

Proposition 2.8.4. Volume type $k = 0$ if and only if the space signature $\{k_1, ..., k_n\}$ contains at least one zero element $k_i = 0$, $i = \overline{1, n}$.

Conjecture 2.8.6. The type of volume in space $\mathbb{B}^n$ with signature $\{k_1, ..., k_n\}$ is equal to

$$k = \prod_{i=1}^{n} K_i.$$

The Chapter 3 **"Application of Theory"** describes some applications of the theory to different domains of algebric geometry, differential geometry and topology. Section 3.1 **"Algebraic geometry"** describes some applications of the theory to the first domain.

Lemma 3.1.1. Nonzero distance $d(\bar{x}, \bar{y})$ between vectors from semi–Euclidean space ${}^d\mathbb{E}_n^m$ is related to distance $d(x, y)$ between corresponding vectors from homogeneous space $\mathbb{B}^n$ with signature $\left\{0, \frac{p_2}{p_1}, \frac{p_3}{p_2}, ..., \frac{p_n}{p_{n-1}}\right\}$ by the following equality:

$$d(x, y) = \begin{cases} d(\bar{x}, \bar{y}), & d(\bar{x}, \bar{y}) \in \mathbb{R}, \\ -i\, d(\bar{x}, \bar{y}), & d(\bar{x}, \bar{y}) \in i\mathbb{R}. \end{cases}$$

Remark. In the case of semi–Euclidean space with defect $d > 1$, homogeneous space signature and geometry are ambiguous. The space allows dilation transformation of angles, and distances don't define all its metric.

Corollary 3.1.1. Homogeneous space $\mathbb{B}^n$ with signature $\{k_1, ..., k_n\}$ is embedded in homogeneous meta–space $\mathbb{R}^{n+1}$ with signature $\{0, k_1, ..., k_n\}$.

Lemma 3.1.2. Semi–Riemannian space ${}^d\mathbb{V}_n^m$ corresponds (in sense of metric) to homogeneous space with signature $\left\{k_1, \frac{p_2}{p_1}, ..., \frac{p_n}{p_{n-1}}\right\}$, where $k_1 = 0$ for linear, $k_1 = 1$ for positive curved and $k_1 = -1$ for negative curved semi–Riemannian space, and p_i, $i = \overline{1,n}$ are coefficients of its metric tensor bilinear form.

Corollary 3.1.2. For small distances, non–linear homogeneous space with signature $\{\pm 1, k_2, ...k_n\}$ is best approximated by linear space with signature $\{0, k_2, ..., k_n\}$.

Lemma 3.1.3. Crystallographic group of space with signature $\{k_1, ..., k_n\}$, generated by main rotations $\mathfrak{R}_1(\varphi_1), ..., \mathfrak{R}_n(\varphi_n)$, is isomorphic to crystallographic group of dual space with signature $\{k_n, ..., k_1\}$, generated by main rotations $\mathfrak{R}'_1(\varphi_n), ..., \mathfrak{R}'_n(\varphi_1)$.

Remark. Although crystallographic groups of dual spaces are isomorphic, the dual spaces are, generally speaking, not isometric, as well as group lattices.

In section 3.2 **"Topology"** we present several results of the theory with effects in topology. The definition of separability of points on a line makes difference between points of elliptic line (non–separable) and Euclidean and hyperbolic ones (separable). For better distinction of the last two cases, define differently the separability of points on a line.

Definition 3.2.1. (Separable and non–separable points). We call points on a line *non–separable*, if all points on this line are connectable with any point on metaplane. We call points on a line *separable*, if for any three points A, B, C on this line and some point D on metaplane, that is connectable with A, C and unconnectable with B, the angle $\angle ADC$ is unmeasurable.

Definition 3.2.3. (Weakly and strongly separable points). We call the points in some line *weakly separable*, if on this line metaplane any point D, being unconnectable with middle point B and connectable with other two points A, C, is connectable with all points from neighborhood of B. We call points on some line *strongly separable*, if in the same conditions any point D is unconnectable not only with B, but also with all points from some its neighborhood.

In these definitions, the points on elliptic line are non-separable, the points on parabolic line are weakly separable, and the points on hyperbolic line are strongly separable.

The definition of neighborhood for spaces with the dimension greater than 1 does not satisfy the duality principle.

Definition 3.2.4 (Generalized neighborhood). We call a *neighborhood* of a point A with defined at it basis $\{a^0 = A, a^1, ..., a^n\}$ in some homogeneous space $\mathbb{B}^n$ to be a set of points, lines and planes of different dimensions such that m-dimensional measure between any of them (point, line or plane) and one from the basis (point, line or plane) of the same dimension m is less than r_m, $m = \overline{0, n-1}$.

This definition is self-dual, that is, the dual definition of the above neighborhood coincides with this definition of neighborhood. The classical definition of neighborhood represents a particular case of this definition, when all angle measures are finite. In this case $r_0 = r, r_i = \frac{\pi}{2}$, $i = \overline{1, n-1}$. This definition is essential in spaces with unbounded measure of some angles.

By changing the neighborhood with its generalization in the axioms of separability of topologic spaces, we obtain generalized axioms of separability. While the majority of homogeneous spaces are not separable, all they are generalized separable.

In the theory of differential manifolds, the homeomorphism between studied space and Euclidean one can be changed to homeomorphism between studied space and homogeneous one, keeping in mind the previous generalizations. Examples of homogeneous manifolds are constructed.

In section 3.3 **"Differential Geometry"**, several useful applications of the theory to this domin are analyzed.

Proposition 3.3.1. Generally speaking, in homogeneous space the geodesic line between points A, B can't be defined as shortest or longest path.

In the **Appendix** we described some possible applications of the theory in domains of theoretic physics (theory of relativity, quantum field theory, string theory and M-theory, AdS/CFT correspondence, cosmology). A hypothetical "non–Euclidean optics" is described.

Finally, we presented the software project GeomSpace (http://sourceforge.net/projects/geomspace/), the system of interactive geometry for homogeneous spaces. This project makes available the results of the analytic geometry of homogeneous spaces.

GENERAL CONCLUSIONS AND RECOMMENDATIONS

The theory of homogeneous spaces raised from:

- the researches related to the problem of the independence of the Euclid's parallel axiom;
- the tendentions in creation of a common theory for different geometries introduced in XIX century by Arthur Cayley and Felix Klein.

The Erlangen Program was aimed to elaborate the overview and the classification of geometric spaces. However this objective was not achieved for rigid geometries of dimension greater than 2, with exception of spaces of constant curvature and spaces needed in physics. That is why the following problem is actual: **to investigate the homogeneous spaces via linear methods applying the concept of signature**. In the Chapter 2, the analytic geometry of homogeneous spaces was constructed. In the Chapter 3 the application of the constructed theory to different areas of mathematics was developed: algebraic geometry, topology and differential geometry, for whose objects it is possible to apply the notion of signature. This permits to formulate the following conclusions:

1. The new notion of signature was introduced. With its aim, the model of homogeneous space with given signature was constructed. That permits the classification of homogeneous spaces based on the concept of signature. Also, for each known homogeneous space (spaces of constant curvature, Galilean, Minkowskii, De Sitter, Anti de Sitter among other spaces) its signature, and its place in the presented classification, were found [12, 13].

2. In dependence on the space signature, the parameterized form of some important axioms were given. Based on them, the formulation and the proof of theorems, parameterized by signature, is also possible in this unified manner. Generalized trigonometric functions were introduced by means of the

signature. These functions make it possible to find universal form of trigonometric equalities, common for each homogeneous space, also intruduced here [13, 15, 18].

3. By the new concept of the group of generalized orthogonal matrix, the isometry group of any homogeneous space was described. In accordance with Felix Klein's concept of geometry, described by him in the well known Erlangen Program, the isometry group of a space determines the geometry of that space [19, 26].

4. Via the new concept of signature, the type (elliptic, parabolic and hyperbolic) of geometric quantities (distances, angles, areas, volumes) was established [14, 16].

5. Based on the new signature concept, the notion of duality of homogeneous spaces was formalized. This leads to the theorem on the isomorphism of crystallographic groups of dual spaces, which was presented here [17].

6. Using the first element of the signature, the topological and metric distinction among elliptic, parabolic and hyperbolic lines was given. This distinction leads to generalization of the neighborhood notion and Hausdorff space notions and makes homogeneous spaces the first class citizens among the metric spaces. This opens the door for homogeneous manifolds study [17].

7. That permits to affirm that the the following problem is completely solved: **to investigate the homogeneous spaces via linear methods applying the concept of signature**.

Different concrete homogeneous spaces were successfully used in mathematics and physics: Euclidean, hyperbolic, Minkowskii, De Sitter. In different courses of mathematics the important role have applications of the geometry of homogeneous spaces. It follows that the analytic geometry of homogeneous spaces has a potential to enlarge and deepen these applications. Our recommandation is to use the analytic geometry of homogeneous spaces

Recommandations: The obtained results and developed methods can be used for:

- further study of different concrete homogeneous spaces;
- research and theoretic modelling of different physical phenomena;
- facultative and optional courses. For this objective, the software application GeomSpace was developed.

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UDC: 514.742.2:514.120/514.140+514.764.21

ANNOTATION

Popa Alexandru, "Analytic geometry of homogeneous spaces", PhD thesis, Chisinau, 2017.

The thesis is written in English and consists of: introduction, three chapters, general conclusions and recommandations, appendix, 210 bibliography titles, 140 pages of main text, 27 figures, 9 algorithms, 5 tables. The obtained results were published in 9 scientific papers.

Keywords: Homogeneous space, Riemannian space, Klein geometry, projective metric, analytic geometry.

Domain of research: Geometry of homogeneous spaces.

Goals and objectives: The goal of the research is to provide a toolchain that can be used to study of homogeneous spaces by means of linear algebra. The objectives of the research are: introduction of the new concept of the space signature, construction of homogeneous space based on signature concept, construction of the model of homogeneous space with given signature, expression of the measurement of different geometric quantities via signature, different applications of the analytic geometry of homogeneous spaces.

Scientific innovation of obtained results:

- Analytic geometry is developed in linear algebra language, even for non–linear spaces.
- One universal theory is developed that uses the elements of space signature as parameters.

Important scientific problem solved: The investigation of the homogeneous spaces with linear methods via concept of the signature.

Theoretical and practical value of the work: Rezultatele prezentate în teză sunt noi, au un caracter teoretic și cu ajutorul conceptului de signatură prezintă o teorie generală a spațiilor omogene.

Implementation of scientific rezults:

- New results can be used in investigation of the problems of differential geometry, in theoretic physics and in other domains where notion of the signature can be applied in the given sense.
- The thesis can be used as the didactic support for optional courses in the university and doctoral studies.

CZU: 514.742.2:514.120/514.140+514.764.21

ADNOTARE

Popa Alexandru, „Geometria analitică a spațiilor omogene", teză de doctor în științe matematice, Chișinău, 2017.

Teza este scrisă în engleză și constă din: introducere, trei capitole, concluzii generale și recomandări, apendice, bibliografie din 210 de titluri, 140 de pagini de text de bază, 27 de figuri, 9 algoritmi, 5 tabele. Rezultatele obținute sunt publicate în 9 lucrări științifice.

Cuvinte-cheie: Spațiu omogen, spațiu Riemanian, geometrie Klein, metrică proiectivă, geometrie analitică.

Domeniul de studii: Geometria spațiilor omogene.

Scopul și obiectivele tezei: Scopul cercetării este să se ofere un instrument, care poate fi folosit pentru a studia spații omogene în limbajul algebrei lineare. Obiectivele sunt: argumentarea conceptului de signatură, pe baza lui, construirea spațiilor omogene, construirea modelului spațiului omogen cu signatura dată, expresia măsurării cantităților geometrice via signatura, aplicațiile geometriei analitice a spațiilor omogene.

Noutatea științifică a rezultatelor obținute:

- Geometria analitică este dezvoltată folosind limbajul algebrei lineere, chiar și pentru spații nelineere.
- Este dezvoltată o teorie universală, în care elementele signaturii spațiului sunt parametri.

Problema științifică importantă soluționată: Cercetarea spațiilor omogene prin metode lineare, aplicând conceptul de signatură.

Semnificația teoretică și valoarea aplicativă a tezei: Rezultatele prezentate în teză sunt noi, au un caracter teoretic și cu ajutorul conceptului de signatură prezintă o teorie generală a spațiilor omogene.

Implementarea rezultatelor:

- Rezultate noi pot fi folosite în investigarea problemelor în geometria diferențială, în fizica teoretică și în alte domenii unde poate fi aplicat conceptul de signatură în sensul dat.
- Teza poate fi folosită în calitate de suport pentru cursurile opționale universitate și postuniversitare.

УДК: 514.742.2:514.120/514.140+514.764.21

АННОТАЦИЯ

Попа Александру, «Аналитическая геометрия однородных пространств», докторская диссертация, Кишинёв, 2017.

Диссертация написана на английском языке и состоит из: введения, трёх глав, общих выводов и рекомендаций, приложения, библиографии из 210 наименований, 140 страниц основного текста, 27 иллюстраций, 9 алгоритмов, 5 таблиц. Результаты исследований опубликованы в 9 научных работах.

Ключевые слова: Однородное пространство, Риманово пространство, геометрия Клейна, проективная метрика, аналитическая геометрия.

Область исследования: Геометрия однородных пространств.

Цели и задачи исследования: Цель исследования — предоставить инструмент для исследования однородных пространств на языке линейной алгебры. Задачи исследования: введение сигнатуры, построение однородного пространства с его помощью, построение модели однородного пространства по сигнатуре, выражение геометрических мер через сигнатуру, приложения аналитической геометрии однородных пространств.

Научная новизна и оригинальность:

- Аналитическая геометрия построена на языке линейной алгебры, даже для нелинейных пространств.
- Разработана одна универсальная теория, в которой элементы сигнатуры пространства являются параметрами.

Решенная научная проблема: Исследование однородных пространств с помощью линейных методов, посредством концепцию сигнатуры.

Теоретическая и прикладная значимость: результаты данной работы новы, имеют теоретический характер, и с помощью концептии сигнатуры предоставляют общую теорию однородных пространств.

Внедрение научных результатов:

- Новые результаты можно использовать в рассмотрении задач дифференциальной геометрии, в физике и других областей, в которых имеет смысл понятие сигнатуры в указанном смысле.
- Диссертацию можно использовать в качестве учебного руководства в факультативных курсах в университете и аспирантуре.

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ANALYTIC GEOMETRY OF HOMOGENEOUS SPACES

111.04 — GEOMETRY AND TOPOLOGY

Synopsis of Doctor Thesis in Mathematical Sciences

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