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INTEGRABILITY OF CUBIC DIFFERENTIAL SYSTEMS WITH INVARIANT STRAIGHT LINES AND INVARIANT CUBICS

111.02. DIFFERENTIAL EQUATIONS

Summary of PhD Thesis in Mathematics

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The defense of the Thesis will take place on December 20, 2019 at 15.00, in the meeting of the Doctoral Committee at the "Dimitrie Cantemir" State University (multipurpose hall, Hincesti 54 Street, Chisinau).

PhD Thesis and Summary can be accessed at the Central Scientific Library "Andrei Lupan", on ANACEC web page (www.cnaa.md) and on USDC web page (www.usdc.md).

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CONCEPTUAL BENCHMARKS OF THE RESEARCH

The Thesis is referred to the qualitative theory of differential equations. The integrability of cubic differential systems, with a singular point of a center or a focus type, having two invariant straight lines and an irreducible invariant cubic is studied.

Actual research status. Modern science has shown that the study of phenomena from nature involves the creation of mathematical models which cover their main characteristics by form. This has led to the fact that the most suitable model for evolutive phenomena can be represented by a system of differential equations. Thus, in the second half of the 19th century the foundations of the modern theory of stability have been laid by works of the Russian mathematician A.M.Lyapunov, which defined in his PhD thesis (1892) the basic concepts of stability. Important results in this direction have been obtained by H. Poincare and J. Maxwell in the study of stability of motion of the celestial bodies. A qualitative leap of differential equations represents the 20th century, by introducing new methods: the method of topological degrees, the theory of bifurcations, etc.

One of the most important problem in the qualitative theory of differential equations is the cyclicity problem, called the local 16th Hilbert problem which deals with the estimation of the number of limit cycles, i.e. isolated periodic solutions, that can bifurcate from a singular point of a center or a focus type, when the coefficients of the differential system are perturbed by an arbitrarily small amount. This problem is part of the still unsolved 16th Hilbert problem given by Hilbert [22] at the beginning of the past century. An important step in solving the cyclicity problem is the problem of distinguishing between a center and a focus, called *the problem of the center*.

The study of the problem of the center was initiated in the works of Poincaré [29] and Lyapunov [28]. Using their proposed methods, the presence of a center can be established by solving an infinite system of polynomial equations whose variables are parameters of the system of differential equations. The polynomial equations are called the center conditions and the polynomials - *the focus quantitis* (or *the Lyapunov quantities*).

By Hilbert's basis theorem, an infinite set of conditions is equivalent to a finite one. There is a finite number of focus quantities such that their vanishing implies the vanishing of all of them. It is only necessary to find a finite number of Lyapunov quantities, though in any given case it is not known a priori how many are required.

Although it dates from the end of the 19th century, the problem of the center is completely solved only for: quadratic systems (Dulac, Kapteyn, Frommer, Saharnikov, Sibirschi, Malkin, Schlomiuk, Żoladek); cubic symmetric systems (Malkin, Sibirschi, Rousseau, Schlomiuk, Żoladek); the Kukles system (Kukles, Cherkas, Christopher, Lloyd, Şubă, Schlomiuk, Sadovskii); the Lotka-Volterra complex quartic system (Ferčec, Giné, Romanovski); the Lotka-Volterra complex quintic system (Gine and Romanovski) etc.

The phase portraits of all quadratic systems having a center were obtaned by Vulpe in [40] and the center conditions for cubic symmetric systems were expressed in terms of algebraic invariants by Sibirschi [37].

The problem of determining when a cubic differential system, containing both quadratic and cubic terms, has a center at a singular point is not completely solved. Necessary and sufficient conditions for a singular point with purely imaginary eigenvalues to be a center were obtained in some particular cases: for cubic systems with degenerate infinity (Kompel, Şubă, Chavarriga, Giné), for cubic systems of a special form containing a certain number of parameters (Daniliuk, Şubă, Romanovski, Lloyd), for a nine-parameter cubic system that can be reduced to a Liénard type system (Bondar, Sadovskii, Shcheglova), for cubic systems with four invariant straight lines and with three invariant straight lines (Şubă, Cozma), for cubic systems with two invariant straight lines and one invariant conic (Cozma), for cubic systems which have a first integral of the form $F_1^{\beta_1} F_2^{\beta_2} = C$ (Baltag), where F_1 and F_2 are polynomials of degree four and six, respectively.

The problem of the center was studied for some classes of polynomial differential systems in the monographs of Sadovskii [34], Romanovski and Shafer [33], Christopher and Li [6], Cozma [13], Zhang [41], Popa and Pricop [30]. The solution to the generalized problem of the center (Ciobanu [8]) was obtained by Popa and Pricop in [31].

The problem of cyclicity of a singular point of a center or a focus type was investigated for some classes of cubic differential systems by Żołądek [43], Bothmer and Kröker [3], Yu and Han [45], Romanovski and Shafer [33], Levandovskyy, Pfister and Romanovski [25], Gaiko [19], Li, Liu and Yang [26], Ferčec and Mahdi [18], etc.

The problem of integrability for polynomial differential systems with singular points of a center type and with a given number of algebraic solutions was studied in the works of Şubă [38], Kooij and Christopher [24], Chavarriga, Giacomini, Giné [5], Christopher, Llibre, Pantazi and Zhang [7], Giné and Romanovski [21], Cao, Llibre and Zhang [4], Dukaric and Giné [17].

A new approach to the problem of the center was realized in the works of Şubă and Cozma [13, 15, 38] for polynomial differential systems by taking into account simultaneously the invariant algebraic curves, the focus quantities and Darboux integrability. They proposed a new direction in investigating the problem of the center for polynomial differential systems, *the problem of center sequences*: for each fixed number n, $n \geq 3$ find all center sequences of polynomial differential systems of degree n with singular points of a center or a focus type.

In [13] Cozma solved the problem of center sequences for cubic differential systems with: four invariant straight lines, three invariant straight lines, two invariant straight lines and one irreducible invariant conic.

In the Thesis we study the problem of the center for the cubic differential system with a singular point of a center or a focus type having two invariant straight lines $l_1 \equiv 1 + a_1x + b_1y = 0$, $l_2 \equiv 1 + a_2x + b_2y = 0$ and one irreducible invariant cubic $\Phi \equiv x^2 + y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 = 0$. The following two fundamental problems were formulated:

Problem 1. *Determine for cubic differential systems the conditions for the existence of two distinct invariant straight lines and one irreducible invariant cubic.*

Problem 2. *Find all center sequences for cubic differential systems with two invariant straight lines and one irreducible invariant cubic.*

The Problems 1 and 2 are solved in Chapters 2, 3, and 4.

Aim of the research: to determine the center conditions for cubic differential systems with two distinct invariant straight lines and one irreducible invariant cubic.

Objectives of the research:

– to obtain the conditions for the existence of two invariant straight lines and one irreducible invariant cubic for the cubic differential system with a singular point of a center or a focus;

– to study the integrability of the cubic systems with two invariant straight lines and one irreducible invariant cubic;

– to solve the problem of the center for cubic systems with two invariant straight lines and one irreducible invariant cubic;

– to find the cyclicity of a singular point of a center or a focus type for cubic systems with two invariant straight lines and one irreducible invariant cubic.

The research hypothesis. The problem of the center for the cubic differential system, with two invariant lines and one invariant cubic, will be solved if: the efficient relations between invariant algebraic curves, focus quantities and local integrability are established; the method of Darboux integrability is developed; the center sequences for cubic systems with two invariant straight lines and one irreducible invariant cubic are determined.

The formulated objectives have contributed to the foundation of scientific concepts. For the first time, for cubic differential systems will be determined new sets of necessary and sufficient conditions for the existence of a center, which represents an important step in solving of the 16th Hilbert problem on limit cycles.

Novelty and scientific originality. The problem of the center was solved for the cubic differential system with a singular point a fine focus having two invariant straight lines $l_1 = 0,\ l_2 = 0$ and one irreducible invariant cubic $\Phi = 0$. Thus:

– new center conditions for the cubic differential system with two invariant straight lines and one irreducible invariant cubic were obtained;

– the method of Darboux integrability can be applied to prove centers in cubic systems when its algebraic solutions form a bundle of curves or they are in generic position;

– the center sequences for cubic differential systems with two invariant straight lines and one irreducible invariant cubic were found;

– new results in the problem of cyclicity for cubic differential systems with invariant straight lines and irreducible invariant cubic were obtained.

The main scientific problem solved consists in establishing of some efficient relations between invariant algebraic curves, focus quantities and local integrability, which contributed to the development of the Darboux integrability method. This made possible to obtain new sets of necessary and sufficient center conditions for cubic differential systems with two invariant straight lines and one invariant cubic.

Methodology of scientific study. The investigations carried out in the Thesis are based on the methods of qualitative theory of dynamical systems, the methods of algebraic computations, the methods of parametrization of the algebraic curves, the method of Darboux integrability and the method of reversibility.

The theoretical significance of the work: it was elaborated an efficient method in solving the problem of center based on relations between the existence of algebraic invariant curves, focus quantities and Darboux integrability.

The practical value of the work: the results obtained for cubic differential systems concerning the problem of the center and the problem of cyclicity represent an important step in solving the 16th Hilbert problem about limit cycles.

Obtained scientific results to be defended:

– the necessary and sufficient conditions for the cubic differential system with a singular point a fine focus to have two invariant straight lines and one irreducible invariant cubic;

– the center sequences for cubic differential systems with two distinct invariant straight lines $l_1 \equiv 1 + a_1x + b_1y = 0$, $l_2 \equiv 1 + a_2x + b_2y = 0$ and one irreducible invariant cubic $\Phi \equiv x^2 + y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 = 0$:

$$(l_1||l_2,\ \Phi;\ N = 2); \quad (l_1,\ l_2,\ l_1 \nparallel l_2,\ \Phi;\ N = 3);$$

– the necessary and sufficient center conditions for cubic differential systems with two invariant straight lines and one irreducible invariant cubic;

– demonstration of Darboux integrability or reversibility for cubic differential systems with a center having two invariant straight lines and one irreducible invariant cubic.

Implementation of the scientific results. The obtained scientific results can be used:

– in investigations of the integrability problem and the problem of limit cycles for polynomial differential systems;

– in the study of mathematical models which describe some social and natural processes: ecology, immunology, the populations dynamics, epidemiology, plasma physics, laser physics, neural networks and others;

– as support for Master Thesis and some optional university courses for students and master students.

Approval of obtained scientific results. The obtained scientific results to be defended were examined and approved by various research seminars: "Differential Equations and Algebras" of Tiraspol State University (13 December 2016, 5 April 2017, 29 January 2019), Chişinău; "Differential Equations" of the Department of Differential Equations and Systems Analysis of Belorussian State University, Minsk, 17 January 2016.

The obtained scientific results were presented at a number of scientific conferences via invited or contributed talks: International Conference "Mathematics and Information Technologies: Research and Education" (MITRE 2019), June 24–26, 2019, Chişinău; International Conference on Mathematics, Informatics and Information Technologies dedicated to the illustrious scientist Valentin Belousov, April 19 – 21, 2018, Bălţi; International Conference "Modern problems of mathematics and its applications in natural sciences and information technologies", September 17–19, 2018, Chernivtsi, Ukraine; The 26th Conference on Applied and Industrial Mathematics (CAIM 2018), Chişinău, Tehnical University of Moldova, September 20 – 23, 2018; The Fourth Conference of Mathematical Society of the Republic of Moldova, Chişinău, June 28 – July 2, 2017; The 25th Conference on Applied and Industrial Mathematics (CAIM 2017), Iaşi, România, September 14 – 17, 2017; Conferinţa Ştiinţifică Internaţională a Doctoranzilor "Tendinţe contemporane ale dezvoltării ştiinţei: viziuni ale tinerilor cercetători", 15 iunie, 2017, Chişinău; International Conference "Mathematics & Information Technologies: Research and Education" (MITRE 2016), June 23–26, 2016 Chişinău; Conferinţa Ştiinţifică Internaţională a Doctoranzilor "Tendinţe contemporane ale dezvoltării ştiinţei: viziuni ale tinerilor cercetători", 25 mai, 2016, Chişinău; Sesiunea naţională de comunicări ştiinţifice a studenţilor, Universitatea de Stat din Moldova, 21–22 aprilie, 2016, Chişinău.

Research publications. The list of publications on the subject of the Thesis contains 15 publications: 4 scientific reviewed articles published in 3 countries (Moldova, România, Ukraine), 3 articles in collections of reviewed articles, 8 proceedings and abstracts of the international conferences.

Keywords: cubic differential system, invariant algebraic curve, the problem of the center, Darboux integrability, center sequence, the problem of cyclicity.

Thesis's structure. Thesis is written in Romanian on 135 pages of main text and includes Introduction, 4 Chapters, General Conclusions and Recommendations, Bibliography with 150 References, Annotations.

CONTENT OF THE THESIS

In the **Introduction** there are presented the actual status of the conducted research, the motivation for the proposed research, the purpose and objectives of the thesis, the importance and advantages of the conducted scientific investigations, novelty and scientific originality, scientific and research problems solved, the scientific results to be defended as well as approvement of obtained scientific results.

Chapter 1 contains a survey of the most important results in the field of integrability of differential equations related to the purpose and objectives of the Thesis: the local 16th Hilbert problem, the integrability problem for polynomial systems with invariant algebraic curves, the problem of center sequences.

We consider the polynomial system of differential equations of the form

$$\frac{dx}{dt} = P(x,y), \quad \frac{dy}{dt} = Q(x,y), \tag{1}$$

where the dependent variables x and y, and the independent one (the time) t are real, $P(x,y)$ and $Q(x,y)$ are relatively prime polynomials in the ring of real polynomials $\mathbb{R}[x,y]$. Denote by $n = \max\{deg\, P, deg\, Q\}$ the degree of the polynomial system. In particular, when $n = 2$ or $n = 3$, the system (1) will be called a *quadratic system* or a *cubic system.*

Let $M(x_0, y_0)$ be a singular point for (1), i.e. $P(x_0, y_0) = Q(x_0, y_0) = 0$. Without loss of generality we can assume that the singular point $O(x_0, y_0)$ of (1) coincides with the origin of coordinates, i.e. $x_0 = y_0 = 0$. We consider the linearization of (1) at $O(0,0)$:

$$\frac{dx}{dt} = a_{10}x + a_{01}y, \quad \frac{dy}{dt} = b_{10}x + b_{01}y. \tag{2}$$

One of the most important question which is still open for planar systems of differential equations is the following: *under which conditions do the original system (1) and the linearized system (2) have the same qualitative behavior and the same topological structure around a singular point* $O(0,0)$?

This problem has been solved unless if the singular point is of a center or a focus type. If the eigenvalues of the linearized system

$$\lambda^2 - (a_{10} + b_{01})\lambda + a_{10}b_{01} - b_{10}a_{01} = 0$$

have non–zero real parts ($Re\lambda_j \neq 0, j = 1,2$) at $O(0,0)$, then the singular point is hyperbolic and the phase diagrams of the system (1) and its linearization (2) are locally qualitatively the same.

Situation change when the eigenvalues are purely imaginary $\lambda_{1,2} = \pm\beta i$, $\beta \neq 0$, $i^2 = -1$. In this case, the singular point $O(0,0)$ is a center for the linearized system (2) and a center or a focus for the nonlinear system (1).

The singular point $O(0,0)$ of differential system (1) is said to be a *focus* if it has a punctured neighborhood where all the orbits spiral in forward or backward time and is said to be a *center* if it has a punctured neighborhood filled of periodic orbits. Coordinate changes of axes and time rescaling bring the system (1) to one the form

$$\dot{x} = y + \sum_{j=2}^{n} p_j(x,y) \equiv P(x,y), \quad \dot{y} = -x - \sum_{j=2}^{n} q_j(x,y) \equiv Q(x,y), \tag{3}$$

where $p_j(x,y)$ and $q_j(x,y)$ are homogeneous polynomials of degree j. In this case for the singular point $O(0,0)$ of (3) we have $\lambda_{1,2} = \pm i$, $i^2 = -1$. Therefore, it is of a center or a focus type, called by some authors *a weak focus* or *a fine focus* or *a monodromic* singular point. Thus, for polynomial differential systems the problem arises of distinguishing between a center or a focus, called *the problem of the center.*

The problem of the center. Find the necessary and sufficient conditions on the polynomials $P(x,y)$ and $Q(x,y)$ of (3) such that a singular point $O(0,0)$ to be a center.

Although this problem dates from the end of the 19th century, it is completely solved only for: quadratic systems $\dot{x} = y + p_2(x,y),\ \dot{y} = -x + q_2(x,y)$; cubic symmetric systems $\dot{x} = y + p_3(x,y),\ \dot{y} = -x + q_3(x,y)$; Kukles system $\dot{x} = y,\ \dot{y} = -x + q_2(x,y) + q_3(x,y)$ and a few particular cases in families of polynomial systems of higher degree. The problem of determining when a cubic differential system containing both quadratic and cubic terms

$$\dot{x} = y + p_2(x,y) + p_3(x,y), \quad \dot{y} = -x + q_2(x,y) + q_3(x,y), \tag{4}$$

has a center at the singular point $O(0,0)$ is not completely solved.

Over the years, several methods have been developed in solving the problem of the center. Thus, Lyapunov [28] proved that a singular point $O(0,0)$ is a center for (3) if and only if the system has a nonconstant analytic first integral $F(x,y) = C$ in a neighborhood of $O(0,0)$. Also, it is known (Amel'kin and other [1]) that $O(0,0)$ is a center for system (3) if and only if the system has an analytic integrating factor of the form

$$\mu(x,y) = 1 + \textstyle\sum_{k=1}^{\infty} \mu_k(x,y),$$

in the neighborhood of $O(0,0)$, where μ_k are homogeneous polynomials of degree k.

In [42] Żołądek mentioned three general mechanisms for producing centers in polynomial systems: searching for 1) a Darboux first integral or 2) a Darboux–Schwarz–Christoffel first integral or by 3) generating centers by rational reversibility. He claimed that these mechanisms are sufficient for producing all cases of real polynomial differential systems with centers.

The problem of the center for polynomial differential systems is closely related to the problem of local integrability of the systems in some neighborhood of a singular point with purely imaginary eigenvalues. In the Thesis we are interested in the algebraic integrability of the polynomial differential systems, called *the Darboux integrability.* It consists in constructing of a first integral or an integrating factor from the algebraic solutions of the polynomial differential system.

Definition 1.1. *An algebraic curve $\Phi(x, y) = 0$ in $\mathbb{C}^2$ with $\Phi \in \mathbb{C}[x, y]$ is an invariant algebraic curve of a polynomial differential system (1) if the following identity*

$$\frac{\partial \Phi}{\partial x} P(x, y) + \frac{\partial \Phi}{\partial y} Q(x, y) \equiv \Phi(x, y) K(x, y) \tag{5}$$

holds for some polynomial $K(x, y) \in \mathbb{C}[x, y]$ called the cofactor of the invariant algebraic curve $\Phi(x, y) = 0$.

We mention that for an algebraic curve $\Phi(x, y) = 0$ of system (1), of degree $n \geq 2$, we have that $deg\, K_\Phi \leq n - 1$.

Definition 1.2. *We say that the invariant algebraic curve $\Phi(x, y) = 0$ is an algebraic solution of (1) if and only if $\Phi(x, y)$ is an irreducible polynomial in $\mathbb{C}[x, y]$.*

If a real polynomial system has a complex invariant algebraic curve then it also has as invariant the conjugate ones.

Theorem 1.2. *For a real polynomial system (1), $\Phi = 0$ is a complex invariant algebraic curve with cofactor K if and only if $\overline{\Phi} = 0$ is a complex invariant algebraic curve with cofactor $\overline{K}$. Here conjugation of polynomials denotes conjugation of the coefficients of the polynomials.*

In the case of a polynomial differential system (3), with a singular point $O(0, 0)$ of a center or a focus type, the forms of the invariant algebraic curves were determined in [13]:

– an invariant straight line can have one of the following forms

$$1 + Ax + By = 0, \quad A,\, B \in \mathbb{C},\ (A, B) \neq (0, 0) \tag{6}$$

or $x - iy = 0,\ x + iy = 0,\ i^2 = -1$;

– an irreducible invariant conic curve can have the form

$$a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + 1 = 0,$$

where $(a_{20}, a_{11}, a_{02}) \neq 0,\ \ a_{20}, a_{11}, a_{02}, a_{10}, a_{01} \in \mathbb{C}$;

– an irreducible invariant cubic curve can have one of the following forms

$$a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + 1 = 0$$

or

$$a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + x^2 + y^2 = 0, \tag{7}$$

where $(a_{30}, a_{21}, a_{12}, a_{03}) \neq 0,\ \ a_{ij} \in \mathbb{C}$.

The calculation of the invariant algebraic curves in the coefficients of a polynomial system is a very hard and cumbersome problem. In some cases it is an unrealistic problem because in general we don't have any evidence about the number of algebraic curves and about the degree of a curve (Giné [20]).

Open problem 1. *What is the maximum number $\alpha(n)$ of algebraic invariant curves in the set of all polynomial differential systems of degree $n > 1$ having finitely many invariant algebraic curves* ?

Open problem 2. *Give a method to find an upper bound M to the degree of the algebraic solutions for a fixed polynomial system of degree $n \geq 2$.*

Over the years, more attention has been given to polynomial differential systems that have invariant algebraic curves. To their study are dedicated a large number of scientific papers. Invariant straight lines were used in studying: quadratic systems by Bautin, Drujkova, Sibirschi, Popa, Schlomiuk, Vulpe; cubic systems by Ljubimova, Kooij, Şubă, Cozma, Sadovskii, Lloyd, Romanovski, Rousseau, Schlomiuk, Puţuntică, Ushkho, Repeşco, Bujac, Vacaraş.

Invariant conics were used in investigating quadratic and cubic system by Cherkas, Schlomiuk, Christopher, Llibre, Oliveira, Cozma, Giné, Vulpe, Sáez, Szánto, Rezende; invariant cubics were considered in studying quadratic and cubic system by Evdokimenko, Cherkas, Županović, Garcia, Cozma, Dascalescu.

In the Thesis, there were obtained the conditions on the coefficient of cubic differential system (4), under which the system has two distinct invariant straight lines of the (6) and one irreducible invariant cubic of the (7). We will consider a first integral (an integrating factor) of the system (3) consisting of invariant algebraic curves.

Definition 1.5. *Suppose the curves $\Phi_j = 0$, $j = 1, \ldots, q$ are invariant algebraic curves of (3) from $\mathbb{C}[x, y]$. A first integral (an integrating factor) of system (3) of the form*

$$\Phi_1^{\alpha_1}\Phi_2^{\alpha_2}\ldots\Phi_q^{\alpha_q} = C \quad \left(\mu = \Phi_1^{\alpha_1}\Phi_2^{\alpha_2}\ldots\Phi_q^{\alpha_q}\right), \tag{8}$$

where α_j, $j = 1, \ldots, q$ are complex numbers, not all zero, is called a Darboux first integral (a Darboux integrating factor).

The method of integration of polynomial differential systems by using invariant algebraic curves was elaborated by Darboux [16]. He proposed, for the first time, a first integral (an integrating factor) of the differential systems having invariant algebraic curves to be constructed in the form (8). If a differential system (3) has a first integral (an integrating factor) of the Darboux form, with $\Phi_j(x, y) = 0$ invariant algebraic curves, then we say that the system is *Darboux integrable.*

Open problem 3. *What are the connections between the possible degrees and the numbers of invariant algebraic curves of a polynomial system of degree n and the existence and the type of its first integral?*

A partial answer to this problem was given by Darboux

Theorem 1.7. *Suppose system (1) has q distinct irreducible invariant algebraic curves $\Phi_j = 0$, $j = 1, \ldots, q$. If $q \geq \frac{1}{2}n(n+1)$, then either we have a Darboux first integral or a Darboux integrating factor.*

Many mathematicians have successfully used the Darboux method of integrability in solving the problem of the center for some classes of polynomial systems: Schlomiuk [35] proved that the method of Darboux can be uniformly applied to prove centers in all cases of the quadratic systems; Şubă and Cozma [14,15] obtained the center conditions for cubic systems with four invariant lines and with three invariant lines [13, 46]; Cozma [10–13] found the center conditions for cubic systems with two invariant straight lines and one irreducible invariant conic; Hill, Lloyd and Pearson [23] determined the center conditions for the Kukles system using algebraic solutions of degree one, two and three.

It is well known from Poincaré [29] that if a differential system with a singular point $O(0,0)$ of a center or a focus type is invariant by the reflection with respect, for example, to the axis $x = 0$ and reversion of time then $O(0,0)$ is a center ($x = 0$ is called the axis of symmetry). Żołądek generalized the notion of symmetry calling it reversibility and was classified the reversible cubic systems with centers.

Definition 1.6. *We say that system (3) is time-reversible if its phase portrait is invariant under reflection with respect to a line and change in the direction of time (reversal of the sense of every trajectory).*

An algorithm for finding all time-reversible polynomial systems was proposed by Romanovski [32] and the relation between time-reversibility and the center-focus problem was studied by Teixeira and Jiazhong [39]. Some bilinear transformations, which transform a given system to one which is symmetric in a line that allowed to obtain the center conditions for some cubic systems were studied by Llyod and Pearson [27], Cozma [9].

Another method for solving the problem of the center is to use the Lyapunov function. There exists a formal power series $F(x,y) = \sum F_j(x,y)$ such that the rate of change of $F(x,y)$ along trajectories of (3) is a linear combination of the polynomials $\{(x^2+y^2)^j\}_{j=2}^{\infty}$:

$$\frac{dF}{dt} = \sum_{j=2}^{\infty} L_{j-1}(x^2 + y^2)^j.$$

Quantities L_j, $j = \overline{1,\infty}$ are polynomials with respect to the coefficients of system (3) called to be the Lyapunov quantities (Amel'kin, Lukashevich and Sadovskii [1], Şubă [38]).

If $L_1 = L_2 = \cdots = L_{m-1} = 0$ and $L_m \neq 0$, then a singular point $O(0,0)$ is a fine focus (weak focus) of order m. At most m small amplitude limit cycles can be bifurcated from $O(0,0)$ by perturbing the coefficients of (3).

Theorem 1.9. *The origin $O(0,0)$ is a center for (3) if and only if $L_j = 0$, $j = \overline{1,\infty}$.*

Thus, the presence of a center can be established by solving an infinite system of polynomial equations whose variables are parameters of the system of differential equations. By Hilbert's basis theorem, an infinite set of conditions is equivalent to a finite one, that is there exists a natural number N such that the infinite system $L_k = 0$, $k = \overline{1,\infty}$ is equivalent with a finite system $L_k = 0$, $k = \overline{1,N}$.

It is only necessary to find a finite number of Lyapunov quantities, though in any given case it is not known a priori how many are required. The vanishing of these Lyapunov quantities yields an algebraic set called the *center variety*, in the parameter space which is the space of coefficients of the vector fields. The number N is known for quadratic systems $N = 3$ (Bautin [2]) and for cubic symmetric systems $N = 5$ (Sibirschi [36], Żołądek [43]). For polynomial differential system (3) we have the following problem:

Open problem 4. *For any degree n ($n \geq 3$) of the polynomial system (3), to find the smallest natural number $N = N(n)$ that vanishing the first N Lyapunov (focus) quantities implies the existence of a center at $O(0,0)$.*

A solution to the Open Problem 4, which allows to solve the generalized problem of the center (Ciobanu [8]), was given by Popa and Pricop. In [31] they obtained an estimation for the maximal number of algebraically independent focus quantities used in solving of the center-focus problem for a given polynomial differential system of degree n.

In [13] Cozma determined the number N for cubic differential system (4) assuming that the system has invariant straight lines and irreducible invariant conics. Thus, $N = 2$ in the case of four invariant lines; $N = 7$ in the case of three invariant lines and $N = 4$ in the case of two invariant straight lines and one invariant conic.

Let the polynomial differential system (3) have M invariant algebraic curves $\Phi_k(x, y) = 0$, $k = 1, \ldots, M$, where $M < n(n + 1)/2$. Under these conditions, taking into account Open Problem 4, determine the smallest natural number N such that vanishing the first N Lyapunov quantities implies the existence of a center at $O(0,0)$.

For the first time this problem was examined by Şubă and Cozma [13, 15, 38]. They proposed a new approach to the problem of the center by simultaneously taking into account the invariant algebraic curves, Lyapunov quantities and Darboux integrability.

Definition 1.7. *We say that $(\Phi_k,\ k = \overline{1, M};\ N)$ is a center sequence for (3), if the existence of M invariant algebraic curves $\Phi_k(x, y) = 0$ and the vanishing of the Lyapunov quantities L_ν, $\nu = \overline{1, N}$ implies the origin $O(0,0)$ to be a center for (3).*

In [13] the problem of center sequences was solved for cubic differential systems (4) with: four invariant lines; three invariant lines; two invariant straight lines and one invariant conic. As the problem of the center for the cubic differential system (4) is not completely solved, in the Thesis we formulate the following two fundamental problems:

Problem 1. *Determine for cubic differential systems the conditions for the existence of two distinct invariant straight lines and one irreducible invariant cubic.*

Problem 2. *Find all center sequences for cubic differential systems with two invariant straight lines and one irreducible invariant cubic.*

Problems 1 and 2 are solved in Chapters 2, 3 and 4.

In **Chapter 2, Cubic systems with two parallel invariant straight lines and one invariant cubic,** the problem of the center is solved for cubic systems with two parallel invariant straight lines and one irreducible invariant cubic ([61], [56], [47]).

Let us consider the cubic differential system (4) written in the form

$$\begin{aligned} \dot{x} &= y + ax^2 + cxy + fy^2 + kx^3 + mx^2y + pxy^2 + ry^3 = P(x,y), \\ \dot{y} &= -(x + gx^2 + dxy + by^2 + sx^3 + qx^2y + nxy^2 + ly^3) = Q(x,y), \end{aligned} \tag{9}$$

where $P(x,y)$ and $Q(x,y)$ are real and coprime polynomials in the variables x and y. The origin $O(0,0)$ is a singular point for (9) of a center or a focus type, i.e. a fine focus.

Suppose the cubic system (9) has two parallel invariant straight lines l_1 and l_2 of the form (6) that are real or complex conjugated ($l_2 = \overline{l_1}$). Then by a rotation of axes of coordinates we can make them parallel to the axis of ordinates (Oy) and the linear part of (9) preserves the form.

Lemma 2.l. *The cubic system (9) has two invariant straight lines parallel to the axis Oy*

$$l_{1,2} \equiv 2 + (c \pm \sqrt{c^2 - 4m})x = 0, \tag{10}$$

if and only if the following coefficient conditions are satisfied

$$a = f = k = p = r = 0, \;\; m(c^2 - 4m) \neq 0. \tag{11}$$

The cofactors of the invariant straight lines are $K_{1,2}(x,y) = y(2mx + c \pm \sqrt{c^2 - 4m})/2$.

For the class of cubic systems (9) with two parallel invariant straight lines the necessary and sufficient conditions for the existence of an irreducible invariant cubic

$$\Phi(x,y) \equiv x^2 + y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 = 0, \tag{12}$$

where $(a_{30}, a_{21}, a_{12}, a_{03}) \neq 0$ and $a_{30}, a_{21}, a_{12}, a_{03} \in \mathbb{R}$, have been obtained in Theorem 2.1.

These conditions are grouped into 9 sets and for each set the problem of the center is solved. The results concerning this problem are included in the following two theorems:

Theorem 2.2. *Suppose cubic system (9) has two parallel invariant straight lines and one irreducible invariant cubic. Then a singular point $O(0,0)$ is a center if and only if the first two Lyapunov quantities vanish.*

Theorem 2.3. *The origin is a center for cubic system (9), with two parallel invariant straight lines (10) and one irreducible invariant cubic (12), if and only if one of the following sets of conditions* (i)–(vii) *holds:*

(i) $a = f = k = p = r = d = l = q = 0$, $s = (-2b^2 - 5bc + 2bg - 3c^2 + 3cg + n)/3$, $m = (-2n)/3$;

(ii) $a = f = k = p = r = l = 0$, $g = [b(b^2 - d^2)]/(2d^2)$, $m = 3(b^2 + d^2)$, $c = -3b$, $n = (-2m)/3$, $q = (bm)/(6d)$, $s = (-b^2m)/(6d^2)$;

(iii) $a = f = k = p = r = 0$, $l = [(5b + 4g - c)d]/9$, $m = 2(b + g)(c - 2b - 2g)$, $q = [(c - g - 2b)d]/3$, $n = [(2b + 4g - c)(5b + 4g - c)]/3$, $s = [(2b + g - c)(c - 2b - 4g)]/9$, $d^2 = (2b + 4g - c)(4b + 2g + c)$;

(iv) $a = f = k = p = r = l = 0,\ g = b + c,\ n = b(b+g),\ m = -2b(b+g),\ s = -b(b+g),$ $q = d(b+g),\ 16b^2 - 3d^2 = 0;$

(v) $a = f = k = p = r = 0,\ c = 2b + 2g,\ l = [(b+c)d]/9,\ q = (dg)/3,\ s = (2g^2)/9,$ $m = (12bg + 8g^2 - d^2)/9,\ n = [2(d^2 - 3bg - 2g^2)]/9;$

(vi) $a = f = k = p = r = l = q = s = 0,\ g = -b,\ c = -2b,\ m = (16b^2 - 3d^2)/16,$ $n = -m;$

(vii) $a = f = k = p = r = d = l = q = 0, m = [(3cu+2bu+3v)(cu-2bu-3v)]/(16u^2), g = b+c-u, n = [-8mu^2-3v(2bu+3cu+3v)]/(8u^2), s = [(8u^2-9cu-6bu-9v)v]/(8u^2).$

The following example shows that the vanishing of the first two Lyapunov quantities is essential for the existence of a center. The cubic system

$$\begin{aligned}
&\dot{x} = y(25c^2x^2 + 108cx + 108)/108,\ c \neq 0,\\
&\dot{y} = -(1296x + 1242cx^2 + 936cxy - 702cy^2 + 297c^2x^3 +\\
&\quad +417c^2x^2y - 221c^2xy^2 - 9c^2y^3)/1296
\end{aligned}$$

has two parallel invariant straight lines $18 + c(9 \pm \sqrt{6}\,)x = 0$ and one irreducible invariant cubic $54(x^2 + y^2) + c(3x + y)^3 = 0$. The origin is a singular point such that $L_1 = 0$ and $L_2 = (-5c^4)/648 \neq 0$. Therefore, the singular point $O(0,0)$ is a focus.

The methods of Darboux integrability and reversibility have played an important role in solving the problem of the center as it is confirmed by the following theorem.

Theorem 2.4. *The cubic system (9) with a center having two parallel invariant straight lines and one irreducible invariant cubic (12) is either Darboux integrable or reversible.*

In **Chapter 3, Cubic systems with a bundle of two invariant straight lines and one invariant cubic,** the problem of the center is solved for cubic differential systems (4) with a bundle of two invariant straight lines and one irreducible invariant cubic ([60], [50], [51], [59], [49], [53], [52], [55]).

Let the cubic system (9) have two invariant straight lines l_1 and l_2, real or complex conjugated ($l_2 = \overline{l_1}$), intersecting at a real singular point (x_0, y_0). By rotating the system of coordinates ($x \to x cos\varphi - y sin\varphi,\ y \to x sin\varphi + y cos\varphi$) and rescaling the axes of coordinates ($x \to \alpha x, y \to \alpha y$), we obtain that $l_1 \cap l_2 = (0,1)$. In this case the invariant straight lines can be written as

$$l_j \equiv 1 + a_j x - y = 0,\ a_j \in \mathbb{C},\ j = 1,2;\ a_2 - a_1 \neq 0. \tag{13}$$

Lemma 2.2. *The cubic system (9) has two distinct invariant straight lines of the form (13) if and only if the following conditions are realized:*

$$\begin{aligned}
&k = (a-1)(a_1 + a_2) + g,\ \ l = -b,\ \ s = (1-a)a_1a_2,\\
&m = -a_1^2 - a_1a_2 - a_2^2 + c(a_1 + a_2) - a + d + 2,\\
&n = a_1a_2(-f-2) - (d+1),\ \ p = (f+2)(a_1 + a_2) + b - c,\\
&q = (a_1 + a_2 - c)a_1a_2 - g,\ \ r = -f - 1,\ (a-1)^2 + (f+2)^2 \neq 0.
\end{aligned} \tag{14}$$

The cofactors of the invariant straight lines (13) look as

$$K_j(x,y) = x + a_j y + ((a-1)a_j + g)x^2 + ((c-a_j)a_j + d + 1)xy + ((f+1)a_j + b)y^2.$$

In Theorems 3.1 and 3.2, for the class of cubic systems (9) with two invariant straight lines (13), there were determined the necessary and sufficient conditions for the existence of an irreducible invariant cubic (12) passing through a singular point $(0,1)$

$$\Phi(x,y) \equiv x^2 + y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 - y^3 = 0 \qquad (15)$$

and forming a bundle with the invariant straight lines (13).

These conditions are grouped into 49 sets and for each set the problem of the center is solved. The results concerning this problem are included in the following two theorems:

Theorem 3.3. *Suppose that $f \neq -2$ and let the cubic system (9) have a bundle of two invariant straight lines (13) and one irreducible invariant cubic (15). Then a singular point $O(0,0)$ is a center if and only if one of the following 16 sets of conditions holds:*

(i) $a = 1,\ b = l = s = 0,\ d = f-1,\ k = g = (ca_1 - 4)/a_1,\ m = (4ca_1 + 3f - 13)/3,$ $n = 2r,\ p = (8 - ca_1 + 4f)/a_1,\ q = -2g,\ r = -(f+1),\ a_1^2 = 3;$

(ii) $a = 1,\ d = -2,\ f = -1,\ k = g,\ l = -b,\ m = (3c^2 - 4b^2 - 4bc - 16)/16,$ $n = -m,\ p = b,\ q = -g,\ r = s = 0;$

(iii) $d = 2a - 3,\ f = (-3)/2,\ g = 2(1-a)(b+c),\ k = (1-a)(2b+c),\ l = -b,$ $m = (9a - 4b^2 - 2bc + 2c^2 - 9)/9,\ n = (18 - 18a + 2b^2 + bc - c^2)/9,\ p = (2b-c)/2,$ $q = 2(a-1)(b+c),\ r = 1/2,\ s = 2(a-1)(2b-c)(b+c)/9;$

(iv) $b = (-1)/5,\ a = -3b,\ c = 18b,\ d = 14b,\ f = 11b,\ g = -2b,\ k = q = 2b,$ $l = -b,\ m = n = -9b,\ p = 21b,\ r = -6b,\ s = 0;$

(v) $b = 1/5,\ a = 3b,\ c = -18b,\ d = -14b,\ f = -11b,\ g = -2b,\ k = q = 2b,$ $l = -b,\ m = n = 9b,\ p = 21b,\ r = 6b,\ s = 0;$

(vi) $b = (-1)/5,\ d = 6b,\ f = 9b,\ p = b,\ r = -4b,\ l = n = -b,\ c = 1/10,\ a = 9c,$ $g = -c,\ m = -3c,\ q = c,\ k = (-3)/20,\ s = 0;$

(vii) $b = 1/5,\ d = -6b,\ f = -9b,\ l = -b,\ n = p = b,\ r = 4b,\ c = (-1)/10,$ $a = -9c,\ g = -c,\ m = 3c,\ q = c,\ k = 3/20,\ s = 0;$

(viii) $a = (-2f-1)/2,\ c = -da_1,\ d = -2r,\ g = (na_1)/2,\ k = 2g,\ n = -2f-3,\ l = -b,$ $m = n(2a_1^2 - 3)/2,\ p = -4g,\ q = -g,\ r = -f-1,\ s = 0,\ a_1 = b/(f+2);$

(ix) $a = (1 - 5f - 2f^2)/(2f+7),\ c = (1-2f)a_2,\ d = 2a + 4f + 3,\ g = c - (a+3)a_2,\ k =$ $4(a-1)a_2 + g,\ l = -b, m = [(11f+21)(2f+3)]/(2f+7), n = [(2f+3)(f-4)]/(2f+7),$ $p = (7f+9)a_2,\ q = 12a_2^3 - 3ca_2^2 - g,\ r = -f-1,\ s = 3(1-a)a_2^2,\ (2f+7)b^2 + (2f+$ $3)(f+2)^2 = 0,\ a_2 = b/(f+2);$

(x) $a = 3/2,\ b = (7c)/6,\ d = -3,\ f = (-3)/2,\ g = (-11c)/6,\ k = -3p,\ l = -b,$ $m = (-41)/6,\ n = 9/2,\ p = c/2,\ q = 7p,\ r = 1/2,\ s = 5/2,\ c^2 - 3 = 0;$

(xi) $a = [2 - u^2(a_2^2 + 2a_2u - 3)]/[2(u^2 + 1)]$, $f = (-a_2^2 - 2a_2u - 4u^2 - 3)/[2(u^2 + 1)]$, $g = (3a_1a_2^2 - 3a_1 - 6a_2^3 + 2b + 2c)/2$, $d = (3 + 2f - 2a - 6a_1a_2 + 9a_2^2)/2$, $2u^2(c + 11b - 4u) + u((2b + c)^2 - 9) + 18b = 0$, $k = (a - 1)(a_1 + a_2) + g$, $l = -b$, $s = (1 - a)a_1a_2$, $m = -a_1^2 - a_1a_2 - a_2^2 + c(a_1 + a_2) - a + d + 2$, $r = -f - 1$, $n = a_1a_2(-f - 2) - (d + 1)$, $p = (f + 2)(a_1 + a_2) + b - c$, $q = (2b - u)a_1a_2 - g$, $a_2 = (c - 2u + 2b)/3$, $a_1 = (4b + 2c - u)/3$;

(xii) $a = 5/6$, $c = 6g$, $d = -3$, $f = (-13)/6$, $g = -5b$, $k = g/3$, $m = 19/54$, $l = -b$, $n = 37/18$, $p = (103b)/3$, $q = (25b)/3$, $r = 7/6$, $s = 1/18$, $108b^2 - 1 = 0$;

(xiii) $a = 2/(5u^2)$, $b = (8 - 5u^2)/(20u)$, $c = (169u^2 - 76)/(10u^3)$, $d = (44 - 105u^2)/(20u^2)$, $f = (4 - 45u^2)/(20u^2)$, $g = (9u^2 - 4)/(2u^3)$, $k = (20 - 49u^2)/(10u^3)$, $l = -b$, $m = (1215u^2 - 508)/(25u^4)$, $n = (45u^2 - 24)/(10u^2)$, $p = 23(4 - 9u^2)/(10u^3)$, $q = 3(8 - 19u^2)/(5u^3)$, $r = -f - 1$, $s = (5u^2 - 2)/(5u^2)$, $5u^4 - 40u^2 + 16 = 0$;

(xiv) $a = 7(11u^2 - 1)/(40u^4)$, $b = (7 - 85u^2)/(200u^5)$, $c = (185u^2 - 19)/(100u^5)$, $d = (5 - 47u^2)/(20u^2)$, $f = (1 - 75u^2)/(40u^2)$, $g = (1 - 3u^2)/(40u^5)$, $k = (9 - 35u^2)/(200u^5)$, $l = -b$, $m = (23 - 229u^2)/(200u^6)$, $n = (105u^2 - 11)/(200u^6)$, $p = (37 - 375u^2)/(200u^5)$, $q = (65u^2 - 11)/(200u^5)$, $r = -f - 1$, $s = (5u^2 + 1)/(200u^6)$, $5u^4 - 10u^2 + 1 = 0$;

(xv) $a = (ha_1)/(h^2 - 1)$, $b = (ha_1)/(h - 1)^2$, $c = [h(14h - 11h^2 - 11)a_1]/[(h^2 + 1)(h - 1)^2]$, $f = (h - 2h^2 - 2)/(h^2 + 1)$, $d = [12(39h^3 - 49h^2 + 28h + 7)]/[(h^2 + 1)(h^2 - 1)^2]$, $g = [24h(27h^3 - 35h^2 + 20h + 5)]/[(h^2 + 1)(1 - h^2)^3]$, $k = (a - 1)(a_1 + a_2) + g$, $l = -b$, $s = (1 - a)a_1a_2$, $m = -a_1^2 - a_1a_2 - a_2^2 + c(a_1 + a_2) - a + d + 2$, $r = -f - 1$, $n = a_1a_2(-f - 2) - (d + 1)$, $p = (f + 2)(a_1 + a_2) + b - c$, $q = (a_1 + a_2 - c)a_1a_2 - g$, $a_2 = (-ha_1)/(h - 1)^2$, $a_1 = 2(h^2 - h + 1)/(1 - h^2)$, $h^4 + 4h^3 - 6h^2 + 4h + 1 = 0$;

(xvi) $a = 2$, $c = [b(h - 2)(1 - 2h)]/(h^2 + 1)$, $f = (h - 2h^2 - 2)/(h^2 + 1)$, $b = (ha_1)/(h - 1)^2$, $d = (-6h^3)/[(h^2 + 1)(h^2 - 1)^2]$, $g = [6h(10h^3 + 3h^2 + 6h - 3)]/[(h^2 + 1)(h^2 - 1)^3]$, $k = (a - 1)(a_1 + a_2) + g$, $l = -b$, $s = (1 - a)a_1a_2$, $m = -a_1^2 - a_1a_2 - a_2^2 + c(a_1 + a_2) - a + d + 2$, $n = a_1a_2(-f - 2) - (d + 1)$, $r = -f - 1$, $p = (f + 2)(a_1 + a_2) + b - c$, $q = (a_1 + a_2 - c)a_1a_2 - g$, $a_2 = (-ha_1)/(h - 1)^2$, $a_1 = 2(h^2 - h + 1)/(1 - h^2)$, $h^4 - 2h^3 - 2h + 1 = 0$.

Theorem 3.6. *Suppose that $f = -2$ and let the cubic system (9) have a bundle of two invariant straight lines (13) and one irreducible invariant cubic (15). Then a singular point $O(0, 0)$ is a center if and only if one of the following 8 sets of conditions is realized:*

(i) $a = (12u^2 - u^4 - 3)/(8u^2)$, $b = (4u^2 - u^4 - 3)/(8u)$, $c = (u^4 + 16u^2 - 17)/(8u)$, $d = (u^2 - 4u^4 - 3)/(4u^2)$, $f = -2$, $g = (3u^6 - 5u^4 + 5u^2 - 3)/(16u^3)$, $k = (6u^6 - u^8 + 24u^4 - 38u^2 + 9)/(64u^3)$, $l = -b$, $m = (5u^6 + 7u^4 - 65u^2 + 29)/(32u^2)$, $n = -d - 1$, $p = b - c$, $r = 1$, $q = (72u^4 - 3u^8 - 22u^6 - 74u^2 + 27)/(64u^3)$, $s = (u^{10} + u^8 - 26u^6 + 54u^4 - 39u^2 + 9)/(128u^4)$;

(ii) $c = [2b(7a-6)]/(3a-2)$, $d = 2(a-1)$, $f = -2$, $g = b(1-a)$, $k = b(a-1)$, $l = -b$, $m = (4-5a)/2$, $n = 1-2a$, $p = b-c$, $q = [b(7a^2-9a+2)]/(3a-2)$, $r = 1$, $s = (a^2-a)/2$, $(3a-2)^2 + 16(a-1)b^2 = 0$;

(iii) $a = (8b-bu^2-2u)/(8b-2u)$, $d = (24b^2-bu(u^2+18)+3u^2)/(4bu-u^2)$, $c = [4b^2(u^2+8)-bu(7u^2+16)+u^2(u^2+2)]/[u^2(u-4b)]$, $f = -2$, $l = -b$, $g = [(8b+u^2-2u)(8b-u^2-2u)(3u^2+4)]/[32u^2(u-4b)]$, $r = 1$, $k = (a-1)(a_1+a_2)+g$, $s = (1-a)a_1a_2$, $q = (a_1+a_2-c)a_1a_2-g$, $m = -a_1^2-a_1a_2-a_2^2+c(a_1+a_2)-a+d+2$, $n = -d-1$, $p = b-c$, $a_1 = (3u^3-4bu^2-16b+4u)/(4u^2)$, $a_2 = (u^4-64b^2+32bu-4u^2)/[4u^2(4b-u)]$;

(iv) $a = (108-u^2)/72$, $b = (u^3-36u)/432$, $f = -2$, $c = (2592-u^4-252u^2)/(432u)$, $d = (-5u^2-36)/72$, $n = -d-1$, $g = (432-u^4+24u^2)/(288u)$, $k = (u^6-3888u^2+93312)/(31104u)$, $m = [(u^4+81u^2-324)(u^2-36)]/(1296u^2)$, $q = [(u^4+168u^2-432)(u^2-36)]/(20736u)$, $s = [(u^2+108)(u^2-36)^2]/373248$, $l = -b$, $p = b-c$, $r = 1$;

(v) $a = (3-2a_1a_2-a_2^2)/2$, $b = l = 0$, $c = 2a_1+3a_2$, $d = 2a-5$, $f = -2$, $g = a_1(3a_2^2+1)/2$, $k = (a_2-2a_1^2a_2+2a_1-a_2^3)/2$, $m = (2a_1^2+6a_1a_2+3a_2^2-3)/2$, $n = -d-1$, $p = b-c$, $q = a_1(-2a_1a_2-7a_2^2-1)/2$, $r = 1$, $s = (1-a)a_1a_2$;

(vi) $a = a_1(h-2a_1)$, $b = (a_1h-a_1^2-1)/h$, $c = 2(a_1^2+2ha_1-h^2+1)/h$, $d = 2a-2$, $f = -2$, $g = (6a_1^3h-3a_1^2h^2+2a_1^2+4a_1h-h^2+2)/(2h)$, $k = (2a_1^3h-8a_1^4+5a_1^2h^2-10a_1^2-2a_1h^3+4a_1h+h^2-2)/(2h)$, $r = 1$, $m = (6a_1^3+a_1^2h-4a_1h^2+6a_1+h^3-2h)/h$, $n = -d-1$, $p = b-c$, $q = (9a_1^2h^2-8a_1^4-6a_1^3h-10a_1^2-2a_1h^3+h^2-2)/(2h)$, $l = -b$, $a_{12} = 3a_1-h$, $s = [a_1(4a_1^4-3a_1^2h^2+6a_1^2+a_1h^3-a_1h-h^2+2)]/h$;

(vii) $a = [a_2(2a_2-2b-c)]/2$, $d = 2(a-1)$, $f = -2$, $g = [6a_2(1-a)-2b+c]/4$, $k = (2a_2-2aa_2-4ab+2ac+2b-c)/4$, $l = -b$, $m = (c^2-4b^2)/4$, $n = 1-2a$, $p = b-c$, $q = [6(a-1)a_2-4ab+2ac+2b-c]/4$, $r = 1$, $s = [a_2(2ab-ac-2b+c)]/2$;

(viii) $a = [a_1(a_2^2+1)]/(a_1-a_2)$, $b = (a_1^2-2a_1a_2-1)/[2(a_1-a_2)]$, $f = -2$, $c = (a_1^2-2a_2^2-1)/(a_1-a_2)$, $d = [2a_2(a_1a_2+1)]/(a_1-a_2)$, $l = -b$, $g = [a_2(3a_1^2a_2+2a_1+a_2)]/[2(a_2-a_1)]$, $k = (a-1)(a_1+a_2)+g$, $s = (1-a)a_1a_2$, $m = -a_1^2-a_1a_2-a_2^2+c(a_1+a_2)-a+d+2$, $r = 1$, $n = -d-1$, $p = b-c$, $q = (a_1+a_2-c)a_1a_2-g$.

Theorem 3.7. *The origin is a center for cubic system (9), with a bundle of two invariant straight lines and one irreducible invariant cubic, if and only if the first three Lyapunov quantities at this point vanish.*

The following example shows that the vanishing of the first three Lyapunov quantities is essential for the existence of a center. The cubic system

$$\dot{x} = (81y + 36x^2 - 81y^2 + 186x^2y - \sqrt{3}(135xy + 4x^3 - 18xy^2)/81,$$
$$\dot{y} = (-27x + 39xy - 5x^3 - 3xy^2 + \sqrt{3}(18x^2 + 9y^2 - 9y^3 - 23xy^2))/27$$

has a bundle of two invariant straight lines $1 - \sqrt{3}x - y = 0$, $1 - \frac{\sqrt{3}}{9}x - y = 0$ and one invariant cubic $9(x^2 + y^2 - y^3) + \sqrt{3}x(x^2 + 9y^2) = 0$. The origin is such that $L_1 = L_2 = 0$ and $L_3 = (-5447680)/3 \neq 0$. Therefore, the singular point $O(0,0)$ is a focus.

The methods of Darboux integrability was applied in solving the problem of the center as it is confirmed by the following theorem.

Theorem 3.8. *The cubic system (9) with a center having a bundle of two invariant straight lines and one irreducible invariant cubic (12) is Darboux integrable.*

In **Chapter 4, Cubic systems with two invariant straight lines and one invariant cubic in generic position,** the problem of the center is solved for cubic differential systems (9) with two invariant straight lines and one irreducible invariant cubic in generic position ([56], [57], [58], [54], [47], [48]).

Let the cubic system (9) have two invariant straight lines l_1 and l_2, real or complex conjugated ($l_2 = \overline{l_1}$), intersecting at a singular point (x_0, y_0). Without loss of generality we can assume that the lines pass through a point $(0, 1)$, i.e. they have the form (13)

$$l_1 \equiv 1 + a_1x - y = 0,\ l_2 \equiv 1 + a_2x - y = 0, a_1, a_2 \in \mathbb{C},\ a_2 - a_1 \neq 0.$$

In this case a_1 and a_2 verifies the following systems of equations:

$$\begin{aligned} &s = a_1(g - k + a_1(a - 1)),\ n = (f + 2)a_1^2 + (b - c - p)a_1 - d - 1, \\ &l = -b,\ r = -f - 1,\ q = -a_1^3 + ca_1^2 + (2 - a + d - m)a_1 - g, \end{aligned} \tag{16}$$

$$\begin{aligned} &F_2 \equiv (a - 1)(a_1 + a_2) + g - k = 0, \\ &F_3 \equiv (f + 2)(a_1 + a_2) + b - c - p = 0, \\ &F_4 \equiv a_1^2 + a_1a_2 + a_2^2 - c(a_1 + a_2) + a - d + m - 2 = 0. \end{aligned} \tag{17}$$

Taking into account the relations (16) and (17), for cubic system (9) there were obtained the necessary and sufficient conditions for the existence of an irreducible invariant cubic $\Phi(x, y) \equiv x^2 + y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 = 0$, where $(a_{30}, a_{21}, a_{12}, a_{03}) \neq 0$, $a_{03} \neq -1$ and $a_{ij} \in \mathbb{R}$.

For cubic system (9) there were obtained 93 sets of necessary and sufficient conditions for the existence of two invariant straight lines and one irreducible invariant cubic in generic position (Theorems 4.1 − 4.4). For each set of conditions the problem of the center is solved. The results concerning this problem are contained in the following two theorems:

Theorem 4.5. *Suppose cubic system (9) has two invariant straight lines and one irreducible invariant cubic in generic position. Then a singular point $O(0, 0)$ is a center if and only if the first three Lyapunov quantities vanish.*

Theorem 4.6. *The origin is a center for cubic system (9), with two invariant straight lines (13) and one irreducible invariant cubic (12) in generic position, if and only if one of the sets of conditions* (i)–(xi) *holds:*

(i) $a = k = r = 0,\ d = f = -1,\ g = (3c - b)/3,\ l = -b,\ m = [2(-bc - 2)]/3, n = bc + 2,$ $p = (2b)/3,\ q = b,\ s = -bc - 2,\ b^2 = 3;$

(ii) $a = b^2 + 1,\ c = r = 0,\ d = 2(b^2 - 1),\ f = -1,\ g = b(3b^2 + 1),\ k = b(b^2 + 1),$ $l = -b,\ m = -b^2,\ n = -4b^2,\ p = -b,\ q = b(-7b^2 - 3),\ s = b^2(-2b^2 - 1);$

(iii) $a = [(3v-1)v]/(3v^2+1)$, $b = \sqrt{3}(1-v^2)/[(3v^2+1)(3v+1)]$, $d = (-15v^3 - 3v^2 - 13v - 1)/[(3v^2+1)(3v+1)]$, $f = (-6v^2+v-1)/(3v^2+1)$, $g = [33v^2 - 1 + \sqrt{3}(3v^2+1)(3v+1)c]/[\sqrt{3}(3v^2+1)(3v+1)]$, $k = (a-1)(a_1+a_2)+g$, $l = -b$, $m = 2 - a_1^2 - a_1a_2 - a_2^2 + c(a_1+a_2) - a + d$, $n = a_1a_2(-f-2) - (d+1)$, $p = (f+2)(a_1+a_2)+b-c$, $q = a_1^2a_2 + a_1a_2^2 - ca_1a_2 - g$, $r = -(f+1)$, $s = a_1a_2(1-a)$, $a_1 = [3v^2+6v-1+\sqrt{3}(c-a_2)(3v^2+1)]/[\sqrt{3}(3v^2+1)]$, $3(3v^2+1)(3v+1)^2(a_2^2-ca_2) + 6(15v^2+1)(v-1) - \sqrt{3}(3v+1)((3v^2+6v-1)(3v+1)a_2 - 3c(3v^2+1)(v-1)) = 0$;

(iv) $a = [(3v+1)v]/(3v^2+1)$, $b = \sqrt{3}(1-v^2)/[(3v^2+1)(3v-1)]$, $d = (-15v^3 + 3v^2 - 13v + 1)/[(3v^2+1)(3v-1)]$, $f = (-6v^2-v-1)/(3v^2+1)$, $g = [33v^2 - 1 + \sqrt{3}(3v^2+1)(3v-1)c]/[\sqrt{3}(3v^2+1)(3v-1)]$, $k = (a-1)(a_1+a_2)+g$, $l = -b$, $m = 2 - a_1^2 - a_1a_2 - a_2^2 + c(a_1+a_2) - a + d$, $n = a_1a_2(-f-2) - (d+1)$, $p = (f+2)(a_1+a_2)+b-c$, $q = a_1^2a_2 + a_1a_2^2 - ca_1a_2 - g$, $r = -(f+1)$, $s = a_1a_2(1-a)$, $a_1 = [-3v^2+6v+1) + \sqrt{3}(c-a_2)(3v^2+1)]/[\sqrt{3}(3v^2+1)]$, $3(3v^2+1)(3v-1)^2(a_2^2 - ca_2) - 6(15v^2+1)(v+1) + \sqrt{3}(3v-1)((3v^2-6v-1)(3v-1)a_2 - 3c(3v^2+1)(v+1)) = 0$;

(v) $a = (2f+3t^2+1)/(2t^2)$, $d = 2a-5$, $l = -b$, $g = [(2ct-4f+1)t^2+2f+1]/(2t^3)$, $k = [(2f+1+3t^2)(ct-2f)]/(2t^3)$, $m = (-3t^2+4tcf-8f^2-2f-1)/(2t^2)$, $n = [tc(f+2) - 2f(f+3) + t^2 - 1]/t^2$, $p = [tc(f+1) + 2f(-f-2)]/t$, $q = [(4f - 2tc-1)t^2 + 2tc(2f+1) - 8f^2 - 6f - 1]/2t^3$, $r = -f-1$, $s = [(2f+t^2+1)(ct-2f)]/(2t^4)$, $t = (f+2)/b$;

(vi) $a = (-10fh^2 + 72fh - 5h^2 + 36h + 108t^2)/(72t^2)$, $b = (f+2)(9-h)/(3t)$, $c = (-4fh + 36f + 5h - 36)/(6t)$, $d = (-10fh^2 + 72fh - 5h^2 + 36h - 72t^2)/(36t^2)$, $g = (-2fh^2 - h^2 + 36t^2)(4h-27)/(216t^3)$, $k = [-(86fh^3 - 1152fh^2 + 3888fh + 43h^3 - 576h^2 - 540ht^2 + 1944h + 3888t^2)]/(432t^3)$, $l = -b$, $m = [-(66fh^2 - 1008fh + 3888f + 49h^2 - 612h + 108t^2 + 1944)]/(72t^2)$, $n = (6fh^2 - 42fh + 7h^2 - 48h + 12t^2)/(12t^2)$, $p = (9fh - 72f + 5h - 36)/(6t)$, $q = [-(2fh^2 - 24fh + h^2 - 12h + 12t^2)(4h-27)]/(72t^3)$, $r = -f-1$, $s = [-(10fh^2 - 72fh + 5h^2 - 36h - 36t^2)(4h-27)h]/(1296t^4)$;

(vii) $a = 3c^2+1$, $b = l = 0$, $d = 2(9c^2-2)/3$, $f = (-5)/3$, $g = c(9c^2+1)$, $k = g$, $m = (-2)/3$, $n = (4-45c^2)/9$, $p = -c$, $q = 2c(-9c^2-1)/3$, $r = 2/3$, $s = cg$;

(viii) $a = [(3f+5)^2(f+2) + b^2(3f+4)^2]/[(3f+5)^2(f+2)]$, $c = [b(6f^2+11f+2)]/[(3f+5)(f+2)]$, $d = [2b^2(3f+4)^3 - (f+2)(5f+7)(3f+5)^2]/[(f+2)(3f+4)(3f+5)^2]$, $g = b[3b^2(3f+4)^2 - (2f+3)(3f+5)^2]/[(f+2)(3f+5)^3]$, $k = -b[b^2(3f+4)^3 + (f+2)(2f+3)(3f+5)^2]/[(f+2)^2(3f+5)^3]$, $l = -b$, $m = -[b^2(3f+4)^2(9f^2+22f+12) + 3(f+1)(f+2)^2(2f+3)(3f+5)]/[(f+2)^2(3f+4)^2(3f+5)]$, $n = -[b^2(3f+4)^2(27f^2+80f+60) - 2(f+1)(f+2)(2f+3)(3f+5)^2]/[(f+2)(3f+4)^2(3f+5)^2]$, $p = -[b(9f^2+22f+12)]/[(3f+5)(f+2)]$, $q = -b[b^2(3f+4)^2(27f^2+85f+66) - (f+1)(f+2)(2f+3)(3f+5)^2]/[(3f+5)^3(3f+4)(f+2)^2]$, $r = -f-1$, $s = -b^2[b^2(3f+4)^2(9f+14) + (f+2)(2f+3)(3f+5)^2]/[(f+2)^2(3f+5)^4]$;

(ix) $a = (4b^3 - bc^2 + 4b + 4c)/(4b + 4c)$, $d = [b(2b+c)^2(2b-c) - 8b(b+c)]/[2(2b+c)(b+c)]$, $f = [3c^2 - (2b-c)^2]/(4b^2 - c^2)$, $g = [(3b^2(2b+c)^2 + 4(b+c)^2)(2b-c)^2]/[16(b+c)^3]$, $k = [(2b-c)^2(b(2b+c)^2(b-2c) + 4(b+c)^2)]/[16(b+c)^3]$, $l = -b$, $m = (c^2 - 4b^2)/4$, $n = [-(4b+c)((2b+c)^2(2b-c)b + 2c(b+c))]/[2(2b+c)^2(b+c)]$, $p = -b-c$, $s = [-b(2b-c)^2((2b+c)^2(2b-c)b + 4(b+c)^2)]/[16(b+c)^3]$, $r = -f-1$, $q = [((14b^2 + 7bc + 2c^2)(2b+c)^2(2b-c)b + 4(12b^2 + 4bc + c^2)(b+c)^2)(c-2b)]/[16(2b+c)(b+c)^3]$;

(x) $a = (9f + p^2 + 18)/[9(f+2)]$, $b = [p(-f-3)]/(3f)$, $c = [p(3-2f)]/(3f)$, $d = (-63f^2 + 2fp^2 - 207f - 162)/[9f(f+2)]$, $g = [p(54f^3 - f^2p^2 + 297f^2 + 540f + 324)]/[27f(f+2)^3]$, $k = [p(p^2(f^2 + 10f + 12) + 27(2f+3)(f+2)^2)]/[27f(f+2)^3]$, $l = -b$, $m = -(18f^3 + f^2p^2 + 81f^2 + 3fp^2 + 117f + 54)/[3f^2(f+2)]$, $n = (36f^3 - f^2p^2 + 162f^2 + 234f + 108)/[3f^2(f+2)]$, $q = [-p(f+1)(27(2f+3)(f+2)^2 - f^2p^2)]/[9f^2(f+2)^3]$, $r = -f-1$, $s = [p^2(27(2f+3)(f+2)^2 - f^2p^2)]/[81f^2(f+2)^3]$;

(xi) $a = 1$, $c = -2b$, $f = -1$, $g = -b$, $k = -b$, $l = -b$, $m = (16b^2 - 3d^2 + 4d + 4)/16$, $n = -m$, $p = b$, $q = b$, $r = s = 0$.

The method of Darboux integrability had an important role in solving the problem of the center as it is confirmed by the following theorem.

Theorem 4.7. *The cubic system (9) with a center having two invariant straight lines and one irreducible invariant cubic (12) in generic position is Darboux integrable.*

GENERAL CONCLUSIONS AND RECOMMENDATIONS

In the Thesis, the problem of distinguishing between a center and a focus, called *the problem of the center* is studied for cubic differential systems. This problem is an important step in solving of the local 16th Hilbert problem which deals with the estimation of the number of limit cycles that can be produced by bifurcations.

The problem of the center for polynomial differential systems is closely related to the problem of local integrability of the systems in some neighborhood of a singular point with purely imaginary eigenvalues. For these reasons, we are interested in the algebraic integrability of the polynomial systems, called *the Darboux integrability.* It consists in constructing of a first integral or an integrating factor from the algebraic solutions of the polynomial differential system. Darboux proved that this is possible to realize for polynomial systems of degree n, if we have $n(n+1)/2$ invariant algebraic curves.

The application of the method of Darboux to prove centers in all cases of the quadratic systems was firstly proved by Schlomiuk [35] and for the cubic systems with two invariant straight lines and one invariant conic was shown by Cozma [13]. A natural question is: how to solve the problem of the center for polynomial differential systems having a fewer number of invariant algebraic curves than $n(n+1)/2$, in particular, for cubic differential systems with two invariant straight lines and one invariant cubic?

In the Thesis, the relations between invariant algebraic curves, focus quantities and local integrability are studied, which lead to the following fundamental problems:

Problem 1. *Determine for cubic differential systems the conditions for the existence of two distinct invariant straight lines and one irreducible invariant cubic.*

Problem 2. *Find all center sequences for cubic differential systems with two invariant straight lines and one irreducible invariant cubic.*

The problems formulated have been completely solved. Research carried out in the Thesis is based on the methods of qualitative theory of dynamical systems, the methods of algebraic computations, the methods of parametrization of the algebraic invariant curves, the methods of local integrability. The problem of the center is studied by using two main mechanisms that are developed in the Thesis: Darboux integrability and reversibility.

For the first time, there were obtained the cubic differential systems with a singular point of a center or a focus type having two invariant straight lines and one irreducible invariant cubic. For these classes of cubic systems: the necessary and sufficient conditions for the existence of a center were obtained; the cyclicity of a fine focus was established; the center sequences with two invariant straight lines and one invariant cubic were found.

These results play an important role in the qualitative study of cubic differential systems and allow us to make the following **general conclusions:**

– the cubic systems with two invariant straight lines and one invariant cubic can have at most three small amplitude limit cycles bifurcated from a singular point of a center or a focus type, which is an important result in the study of the problem of cyclicity (Cap. 2, 2.2; Cap. 3, 3.4; Cap. 4, 4.7);

– the cubic systems with a center having two invariant straight lines and one irreducible invariant cubic are Darboux integrable in all cases when the algebraic solutions form a bundle of curves or they are in generic position (Cap. 3, 3.2, 3.4; Cap. 4, 4.7);

– the cubic systems with a center having two parallel invariant straight lines and one irreducible invariant cubic are Darboux integrable or reversible (Cap. 2, 2.3);

– the cyclicity of a singular point of a center or a focus type, in cubic differential systems, depends on the relative position of the invariant algebraic curves (Cap. 3, 3.1, 3.3; Cap. 4, 4.1 – 4.4.).

The main scientific problem solved consists in establishing of some efficient relations between invariant algebraic curves, focus quantities and local integrability, which contributed to the development of the Darboux integrability method. This made possible to obtain new sets of necessary and sufficient center conditions for cubic differential systems with two invariant straight lines and one invariant cubic.

The results obtained for cubic differential systems concerting the problem of the center represent an important step in solving the 16th Hilbert problem about limit cycles.

Based on the conclusions presented above we can **recommend the following:**

– to study the problem of the center for cubic differential systems having algebraic solutions with the sum of degrees equal to a number given a priori;

– to use the algebraic invariants in studying the problem of the center for cubic differential systems with invariant algebraic curves;

– to apply the obtained results in qualitative investigation of cubic differential systems, in studying of some mathematical models which describe social and natural processes;

– to include the obtained results in the programs of the optional university courses for students and master students.

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ADNOTARE

Dascalescu Anatoli, "Integrabilitatea sistemelor diferenţiale cubice cu drepte şi cubice invariante". Teză de doctor în ştiinţe matematice. Chişinău, 2019

Structura tezei: teza constă din introducere, patru capitole, concluzii generale şi recomandări, bibliografie din 150 titluri, 135 pagini de text de bază. Rezultatele obţinute sunt publicate în 15 lucrări ştiinţifice.

Cuvinte-cheie: sistem diferenţial cubic, curbă algebrică invariantă, problema centrului, integrabilitatea Darboux, consecutivitate centrică, problema ciclicităţii.

Domeniul de studiu: teoria calitativă a sistemelor dinamice, integrabilitatea sistemelor diferenţiale polinomiale.

Scopul lucrării: determinarea condiţiilor de existenţă a centrului pentru sistemul diferenţial cubic cu două drepte invariante distincte şi o cubică invariantă ireductibilă.

Obiectivele cercetării: determinarea condiţiilor de existenţă a două drepte invariante şi a unei cubice invariante ireductibile pentru sistemul cubic cu punct singular de tip centru sau focar; studierea integrabilităţii sistemelor; rezolvarea problemei centrului şi problemei ciclicităţii în prezenţa a două drepte invariante şi o cubică invariantă ireductibilă.

Noutatea şi originalitatea ştiinţifică: a fost rezolvată problema centrului pentru sistemul diferenţial cubic cu două drepte invariante şi o cubică invariantă ireductibilă; a fost stabilită ciclicitatea punctului singular de tip centru sau focar; au fost determinate consecutivităţile centrice. A fost demonstrat că orice sistem cubic ce are punct singular de tip centru, două drepte invariante şi o cubică invariantă ireductibilă, este Darboux integrabil sau reversibil.

Problema ştiinţifică importantă soluţionată constă în stabilirea unor relaţii eficiente dintre existenţa curbelor algebrice invariante, mărimile focale şi integrabilitatea locală, ceea ce a contribuit la dezvoltarea metodei de integrabilitate Darboux, fapt ce a permis determinarea unor noi seturi de condiţii necesare şi suficiente de existenţă a centrului pentru sistemele diferenţiale cubice cu două drepte invariante şi o cubică invariantă.

Semnificaţia teoretică a lucrării: a fost dezvoltată metoda de investigare a problemei centrului care se bazează pe relaţiile dintre existenţa curbelor algebrice invariante, mărimile focale şi integrabilitatea Darboux.

Valoarea aplicativă a lucrării: pentru sistemele diferenţiale cubice au fost obţinute rezultate noi ce ţin de problema centrului şi ciclicităţii, care reprezintă o etapă importantă în rezolvarea problemei a 16-a a lui Hilbert despre ciclurile limită.

Implementarea rezultatelor ştiinţifice: rezultatele obţinute în teză pot fi aplicate în investigaţiile problemei integrabilităţii şi a problemei ciclurilor limită pentru sistemele diferenţiale polinomiale; pot servi drept suport pentru tezele de master şi unele cursuri opţionale universitare ţinute studenţilor şi masteranzilor; pot fi folosite în studiul unor modele matematice ce descriu procese sociale şi naturale.

АННОТАЦИЯ

Даскалеску Анатолий, "Интегрирование кубических дифференциальных систем с алгебраическими инвариантными кривыми первого и третьего порядка". Диссертация на соискание учёной степени доктора математических наук. Кишинэу, 2019

Структура работы: введение, четыре главы, выводы и рекомендации, библиография из 150 наименований, 135 страниц основного текста. Полученные результаты были опубликованы в 15 научных работах.

Ключевые слова: кубическая дифференциальная система, алгебраическая инвариантная кривая, проблема центра, интегрируемость в смысле Дарбу, центрическая последовательность, проблема цикличности.

Область исследования: качественная теория динамических систем, интегрируемость полиномиальных дифференциальных систем.

Цель работы: определение условий существования центра для кубической дифференциальной системы с особой точкой типа центра или фокуса при наличии двух инвариантных прямых и инвариантной кривой третьего порядка.

Задачи исследования: нахождение условий существования двух инвариантных прямых и инвариантной кривой третьего порядка в кубической дифференциальной системы с особой точкой типа центра или фокуса; определение условий интегрируемости систем; решение проблемы центра и проблемы цикличности для кубических систем с двумя инвариантными прямыми и инвариантной кривой третьего порядка.

Новизна и научная оригинальность: для кубической дифференциальной системы с двумя инвариантными прямыми и инвариантной кривой третьего порядка была решена проблема центра и была установлена цикличность особой точки типа центра или фокуса. Было доказано, что любая кубическая система с особой точкой типа центра при наличии двух инвариантных прямых и инвариантной кривой третьего порядка интегрируема в смысле Дарбу или имеет ось симметрии.

Главная решенная научная задача состоит в установлении эффективных соотношении между алгебраическими инвариантными кривыми, фокусными величинами и локальной интегрируемостью, что способствовало развитию метода интегрируемости в смысле Дарбу, что позволило получить новые необходимые и достаточные условия центра для кубических систем с двумя инвариантными прямыми и инвариантной кривой третьего порядка.

Теоретическая значимость работы: был разработан метод исследования проблемы центра, основанный на соootношениях между алгебраическими инвариантными кривыми, фокусными величинами и интегрируемостью в смысле Дарбу.

Практическая значимость работы: были полученны новые результаты по проблеме центра и проблеме цикличности, которые являются важным шагом в решении 16-й проблемы Гильберта о предельных циклах.

Внедрение научных результатов: полученные результаты могут быть использованы при дальнейшем изучении проблемы интегрируемости и проблемы предельных циклов полиномиальных систем, при разработке тем магистерских работ и некоторых спецкурсов для физико-математических специальностей, при исследовании некоторых математических моделей, описывающих социальные и природные процессы.

ANNOTATION

Dascalescu Anatoli, "Integrability of cubic differential systems with invariant straight lines and invariant cubics". PhD Thesis in Mathematical Sciences. Chişinău, 2019

Thesis structure: introduction, four chapters, general conclusions and recommendations, bibliography of 150 titles, 135 pages of main text. The obtained results were published in 15 scientific papers.

Keywords: cubic differential system, invariant algebraic curve, the problem of the center, Darboux integrability, center sequence, the problem of cyclicity.

Domain of research: qualitative theory of dynamical systems, integrability of polynomial differential systems.

Aim of the research: to determine the center conditions for the cubic differential system with two distinct invariant straight lines and one irreducible invariant cubic.

Objectives of the research: to obtain the conditions for the existence of two invariant straight lines and one irreducible invariant cubic for the cubic differential system with a singular point of a center or a focus; to study the integrability of the systems; to solve the problem of the center and the problem of cyclicity for the cubic systems with two invariant straight lines and one irreducible invariant cubic.

Novelty and scientific originality: the problem of the center, the problem of cyclicity and the problem of center sequences were solved for cubic differential systems with two invariant straight lines and one irreducible invariant cubic. It was proved that every cubic differential system with a center, having two invariant straight lines and one irreducible invariant cubic, is Darboux integrable or reversible.

The main scientific problem solved consists in establishing of some efficient relations between invariant algebraic curves, focus quantities and local integrability, which contributed to the development of the Darboux integrability method. This made possible to obtain new sets of necessary and sufficient center conditions for cubic differential systems with two invariant straight lines and one invariant cubic.

The theoretical significance of the work: it was elaborated an efficient method in solving the problem of center based on relations between the existence of algebraic invariant curves, focus quantities and Darboux integrability.

The practical value of the work: the results obtained for cubic differential systems concerning the problem of the center and the problem of cyclicity represent an important step in solving the 16th Hilbert problem about limit cycles.

Implementation of the scientific results: the obtained results can be applied in investigations of the problem of integrability and the problem of limit cycles for polynomial differential systems; can serve as support for Master Thesis and some optional university courses for students and master students; can be used in the study of mathematical models which describe some social and natural processes.

DASCALESCU ANATOLI

INTEGRABILITY OF CUBIC DIFFERENTIAL SYSTEMS WITH INVARIANT STRAIGHT LINES AND INVARIANT CUBICS

111.02. DIFFERENTIAL EQUATIONS

Summary of the PhD Thesis in Mathematics

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